Hamilton-Jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction

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Abstract

We consider a family of open star-shaped domains Ω^ε of $\mathbb{R}^2$. Roughly speaking, Ω^ε is made of a finite number of non intersecting semi-infinite strips of thickness ε and of a central region whose diameter is of the order of ε, that may be called the junction. When the thickness ε tends to 0, the domains Ω^ε tend to a union of half-lines sharing an endpoint O. This set is termed *network*. We study infinite horizon optimal control problems in which the state is constrained to remain in $\overline{\Omega^\varepsilon}$. In the above mentioned strips the running cost may have a fast variation w.r.t. the transverse coordinate. We pass to the limit as the parameter ε tends to zero, and prove that the value function tends to the solution of a Hamilton-Jacobi equation on the network, which may also be related to an optimal control problem. One difficulty is to find the transmission condition at the junction node O in the limit problem. For passing to the limit, we use the method of the perturbed test-functions of Evans, which requires constructing suitable correctors. This is another difficulty since the domain is unbounded.

1 Introduction

A network (or a graph) is a set of items, referred to as vertices or nodes, with connections between them referred to as edges, see the right part of Figure 1 for an example. In the recent years there has been an increasing interest in the investigation of dynamical system and differential equation on networks, in particular in connection with problem of data transmission and traffic management (see for example Garavello-Piccoli [19], Engel et al [16]).

Nevertheless, the literature on continuous-state and continuous-time control on networks is still scarce: there is the recent article [1], where the authors (including those of the present paper) consider control problems whose dynamics is constrained to a network and related Bellman equations. They introduce a definition of viscosity solution which reduces to the usual one if the network is only composed of two parallel segments entering in a node: while in the interior of an edge one can test the equation with a smooth test-function, the main difficulties arise at the junctions where the network does not have a regular differential structure; at a vertex, a notion of derivative similar to that of Dini's derivative (see for example [9]) is proposed, hence admissible test-functions are the ones which admit derivatives in the directions of the edges sharing the node. With this definition, the intrinsic geodesic distance, fixed one argument, is an admissible test-function with respect to the other argument. The above mentioned notion

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of viscosity solutions is equivalent to the one introduced independently by Imbert, Monneau and Zidani [21] for studying a Hamilton-Jacobi approach to junction problems and traffic flows. This was proved by Camilli and Marchi in [15]. There is also the work by Schieborn and Camilli [25], in which the authors focus on eikonal equations and on a less general notion of viscosity solution.

Both [1] and [21] contain the first comparison and uniqueness results, but in these works the assumptions are somewhat restrictive. More general comparison results have been proved in two very recent papers [2] and [20], which both handle more general transmission conditions at the crosspoints than in [1] and [21]. The paper by Imbert-Monneau [20] contains the most general comparison and uniqueness results: the proof, rather involved, is completely based on partial differential equations arguments and works for bimonotone Hamiltonians (not necessarily convex). In the more particular context of Hamilton-Jacobi equations coming from optimal control problems on networks, a short proof of a general comparison result is proposed in [2], whose authors use arguments from optimal control theory, adapting original ideas that Barles, Briani and Chasseigne, see [12, 11], have used for control problems with discontinuous dynamics and costs. In what follows, we shall use the comparison results of [20], which are summarized and adapted to the present context in § 4.2 below.

The aim of this paper is to study the asymptotic behavior of the value function of an optimal control problem in which the dynamics is constrained to remain in the closure of a bidimensional thin open set Ω_ε converging to a network as the width parameter ε tends to 0. We restrict ourselves to the case of a single junction, i.e. the network has only one crosspoint: the set Ω_ε may be divided into a finite number of strips (which may be called the *roads*) and a *junction zone*, see the left part of Figure 1. Precise assumptions on the class of domains Ω_ε will be made in § 1.1 below. In the roads (whose width is of the order of ε), the running cost may have fast variations in the transverse direction but not in the longitudinal direction. Similarly, in the roads, the state constraints involve only the (fast) transverse variable. This is why the problem may be termed *a singularly perturbed problem*.

In the case when there is no junction, singularly perturbed problems in optimal control have been studied by many authors, see for example Bensoussan [14], Artstein and Gaitsgory [7], Gaitsgory and Leizarowitz [18], Bagagiolo and Bardi [8], Alvarez and Bardi [3, 4], Terrone [26], Alvarez-Bardi-Marchi [6, 5]. The reference [3] is of special interest here since the authors focus on Hamilton-Jacobi equations and use viscosity solutions arguments in order to study the asymptotic behavior of the value function: the method for proving the convergence to the dimension reduced effective problem is based on the perturbed test-function method of Evans [17], which implies the construction of correctors. The correctors are viscosity solution to some first order partial differential equation in the fast variable with state constrained boundary conditions.

The aim of the present paper is to perform the same kind of analysis for the family of the domains Ω_ε displayed on the left of Figure 1 as the width parameter ε tends to 0. The method uses viscosity solutions arguments and is reminiscent of that proposed in [3]. The effective equation far away from a junction is found in a straightforward manner by using the results contained in [3]. The main difficulty consists of finding the transmission conditions at the junction. Naturally, the latter depend on the dynamics and running cost in the junction zone. The convergence of the value function will be proved by using the comparison principle stated in [20]. The main technical point lies in the construction of junction-correctors and in their use in the perturbed test function method. The strategy for the construction of the junction-correctors is reminiscent of the one used by Ishii in [22]. An important difficulty comes from the unboundedness of the domain in which the correctors are defined: indeed, for obtaining bounded correctors, we will have to work in suitable unbounded subdomains obtained by truncating the original one. In turn, since the correctors are defined in subdomains, the method of the perturbed test functions of Evans will have to be suitably modified.

1.1 The setting

1.1.1 The geometry

For simplicity, let us focus on the model case of a star-shaped network with N straight edges, $N > 1$. Let $(e_i)_{i=1,\dots,N}$ be a set of unit vectors in $\mathbb{R}^2$ s.t. $e_j \neq e_k$ if $j \neq k$. Note that $e_j = -e_k$ is possible. We assume that there is at least a pair (j,k), $j \neq k$ s.t. e_j is not aligned with e_k. For any $i = 1,\dots,N$, let $e_i^\perp$ be a unit vector perpendicular to e_i. A given point $z \in \mathbb{R}^2$ is written $z = z_i e_i + z_i^\perp e_i^\perp$. The open half-line $\mathbb{R}_+ e_i$ is denoted G_i, and the star shaped network $\mathcal{G}$ (see Figure 1) is defined by

$$\mathcal{G} = \{O\} \cup \bigcup_{i=1}^N G_i,$$

where O is the origin $O = (0,0)$.
For a radius $\rho > 0$, let the convex polygonal domain W_ρ be defined by

$$W_\rho = \{z \in \mathbb{R}^2 : \ \forall i = 1,\dots,N, \ z \cdot e_i < \rho\}. \tag{1.1}$$

Let Ω be a connected open subset of $\mathbb{R}^2$ with the following properties, see Figure 1:

- Ω contains the origin O and is star-shaped with respect to O
- Ω has a smooth boundary
- There exists a positive radius r_0 such that

  $$\Omega \backslash \overline{W}_{r_0} = \cup_{i=1}^N Z_i,$$

  where W_ρ is given by (1.1) and Z_i is the half-strip

  $$Z_i = \{z = z_i e_i + z_i^\perp e_i^\perp;\ -1 < z_i^\perp < 1;\ z_i > r_0\}. \tag{1.2}$$

  In other words, outside the region $\overline{W}_{r_0}$, Ω coincides with the union of the N non intersecting semi-infinite strips of width 2 and aligned with the vectors e_i, $i = 1,\dots,N$.

Let $\widetilde{K}_0$ be the subset of Ω defined by

$$\widetilde{K}_0 = \Omega \cap W_{r_0} \tag{1.3}$$

where W_ρ is given by (1.1).
The control problems will take place in the set $\Omega_\varepsilon = \varepsilon\Omega$, which can be viewed as a thick version of $\mathcal{G}$. The thickness parameter $\varepsilon > 0$ is bound to tend to zero.

1.1.2 The control problem in Ω_ε

Let A be a compact subset of $\mathbb{R}^2$ and $\ell_\varepsilon : \overline{\Omega}_\varepsilon \times A \to \mathbb{R}$ be a continuous function.
For a given positive number λ and any $z_0 \in \overline{\Omega}_\varepsilon$, we consider the infinite horizon control problem consisting of minimizing the cost functional

$$J_\varepsilon(z_0, \alpha) = \int_0^\infty \ell_\varepsilon(z_\varepsilon(t; z_0, \alpha), \alpha(t)) e^{-\lambda t} dt$$

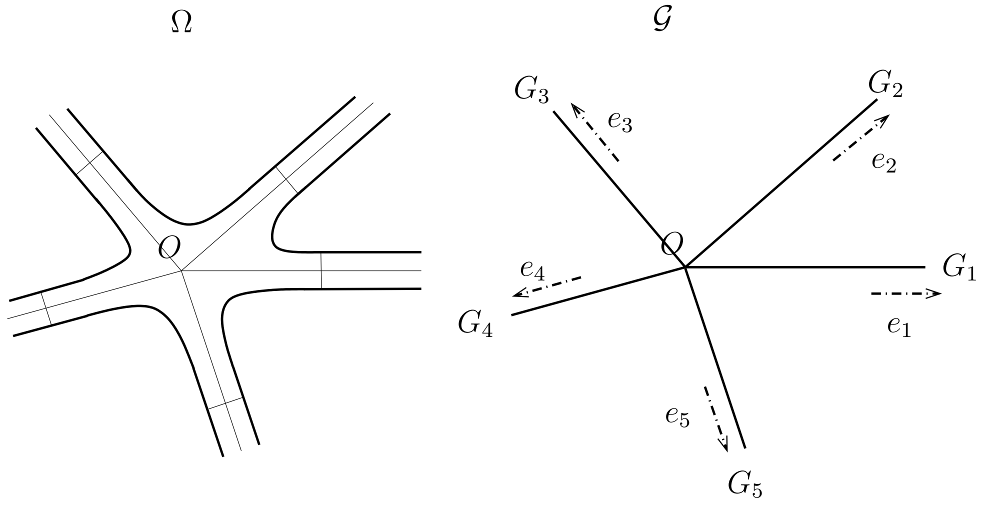

Figure 1: The set Ω and the network $\mathcal{G}$ ($N = 5$)

on the trajectories of the system

$$\dot{z}_\varepsilon(t; z_0, \alpha) = \alpha(t), \quad t > 0, \qquad z_\varepsilon(0; z_0, \alpha) = z_0,$$

where the control function $\alpha : \mathbb{R}_+ \to A$ is measurable and such that the corresponding trajectory satisfies the state constraint $z_\varepsilon(t) \in \overline{\Omega}_\varepsilon$ for all $t \geq 0$. We start by making some assumptions on the structure of the problem.

Assumption 1.1. *There exists a positive constant r such that A contains the ball $B(0, r)$.*

Assumption 1.2.
- *The function ℓ_ε is continuous*
- *It has the following structure: there exist N bounded and continuous functions $\ell_i : [0, +\infty) \times [-1, 1] \times A \to \mathbb{R}$, $i = 1, \dots, N$, and a function $\ell_0 : \overline{\widetilde{K}_0} \times A \to \mathbb{R}$ such that*

$$\begin{aligned} \ell_\varepsilon(z, a) &= \ell_i(z_i - \varepsilon r_0, \frac{z_i^\perp}{\varepsilon}, a) && \text{in } \varepsilon \overline{Z}_i, \\ \ell_\varepsilon(z, a) &= \ell_0(\frac{z}{\varepsilon}, a) && \text{in } \varepsilon \overline{\widetilde{K}_0}. \end{aligned} \tag{1.4}$$

The functions ℓ_i are uniformly continuous in $[0, +\infty) \times [-1, 1]$ uniformly w.r.t. $a \in A$. The function ℓ_0 is uniformly continuous in $\overline{\widetilde{K}_0}$ uniformly w.r.t. $a \in A$.

Assumption 1.3. *Since the function ℓ_0 is bounded, it is not restrictive to assume that ℓ_0 takes nonnegative values (just add a constant to ℓ_0 if necessary).*

Assumption 1.1 is on the controllability of the system. The most important part of Assumption 1.2 says that away from the junction, i.e. for z belonging to the strip $[\varepsilon r_0, +\infty) e_i \times [-\varepsilon, \varepsilon] e_i^\perp$, $1 \leq i \leq N$, the running cost has a fast dependence w.r.t. the transverse coordinate $z_i^\perp$ and a slow dependence w.r.t. the tangential coordinate z_i.

With Assumptions 1.1 and 1.2, it is well known, see e.g. [9] that the value function u_ε of the control problem described above is bounded uniformly with respect to ε, continuous, and is the unique viscosity solution of

$$\lambda u_\varepsilon(z) + H_\varepsilon(z, Du_\varepsilon) \geq 0 \qquad \text{in } \overline{\Omega}_\varepsilon, \tag{1.5}$$

$$\lambda u_\varepsilon(z) + H_\varepsilon(z, Du_\varepsilon) \leq 0 \qquad \text{in } \Omega_\varepsilon, \tag{1.6}$$

where for $z \in \overline{\Omega}_\varepsilon$, $p \in \mathbb{R}^2$, the Hamiltonian $H_\varepsilon(z,p)$ is

$$H_\varepsilon(z,p) = \max_{a\in A}\Big(-p\cdot a - \ell_\varepsilon(z,a)\Big). \tag{1.7}$$

Here $p \cdot a$ denotes the scalar product of p by a.

1.2 Organization of the paper

Here, we propose an informal overview of the main result and of the notions that it requires. Since this paragraph is only meant to help the reader find her/his way in the paper, we do not mean to give full details, and rather refer to the places where the definitions are thoroughly written.

The main result is Theorem 4.2. Its statement is the following: under Assumptions 1.1-1.3 and the further Assumption 2.1, the sequence u_ε converges locally uniformly to the bounded viscosity solution $u : \mathcal{G} \to \mathbb{R}$ of the Hamilton-Jacobi equation on $\mathcal{G}$,

$$\left\{\begin{array}{rcl} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) &=& 0 \quad x = x_i e_i \in G_i, \\ \\ \lambda u(O) + \max\Big(E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0))\Big) &=& 0. \end{array}\right.$$

Let us list the necessary notions for this theorem and the places where they are defined:

1. The effective Hamiltonian $\overline{H}_i$ corresponding to the edge G_i is defined in § 2:

$$\overline{H}_i(x_i,p_i) = \sup_{\mu\in\mathcal{Z}_i}\left(\int_{[-1,1]\times A}\Big(-p_i a_i - \ell_i(x_i,y,a)\Big)d\mu(y,a)\right),$$

   where $\mathcal{Z}_i$ is a compact and convex set of Radon probability measures on $[-1,1]\times A$. This set may be viewed as a set of limiting relaxed controls. A technical assumption, useful for the construction of the junction-correctors, is introduced in § 2.3.

2. The constant E can be viewed as the opposite of an effective cost at the junction: it is defined in § 3. It strongly depends on ℓ_0.

3. The Hamiltonian $\overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0))$ appearing in the effective transmission condition at the junction is defined in § 4.1 : $\overline{H}(O,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is given by

$$\overline{H}(O,p_1,\dots,p_N) \equiv \max_{i=1,\dots,N} \overline{H}_i^+(0,p_i),$$

   where

$$\overline{H}_i^+(0,p_i) = \sup_{\mu\in\mathcal{Z}_i^+}\left(\int_{[-1,1]\times A}\Big(-p_i a_i - \ell_i(0,y,a)\Big)d\mu(y,a)\right),$$

   and

$$\mathcal{Z}_i^+ = \left\{\mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1]\times A} a_i d\mu(y,a) \ge 0\right\}.$$

   The Hamiltonian $\overline{H}(O,\cdot)$ is thus constructed by considering only the controls for which the relaxed dynamics starting from O point in one of the edges of $\mathcal{G}$ (or stay at O).

4. The notion of viscosity solution of the above Hamilton-Jacobi equation has been defined in [1], and comparison results have been proved in [2] and [20]. These notions are reviewed in § 4.2.

The main difficulties for proving Theorem 4.2 are the following:

- to identify the effective constant E: in § 3.1, E is constructed as the limit of a sequence of ergodic constants related to larger and larger bounded subdomains of Ω with running cost ℓ_0.

- to construct correctors that will be used in the perturbed test-function argument of Evans, see [17]. The difficulty lies in the fact that the correctors need to be bounded functions defined in unbounded domains, ideally the whole domain Ω. In general, it will not be possible to define these correctors in the full domain Ω, and we will have to restrict ourselves to suitable unbounded subdomains of Ω; The construction of correctors is done in §5. The proof of convergence is given in § 6.

2 The effective Hamiltonian in the edges

The first step in understanding the asymptotic behavior of the value function u_ε as $\varepsilon \to 0$ is to look at what happens far from the region $\varepsilon \widetilde{K}_0$, i.e. far from the junction. For that, it is possible to rely on existing results. In the whole Section 2, i is an index in $\{1, \dots, N\}$.

2.1 Known facts

Singular perturbations in deterministic control have been studied by many authors from the viewpoint of either control theory or viscosity solutions. In particular, state constrained control problems in thin domains obtained by thickening a smooth manifold have been much studied. The results of Alvarez-Bardi [3] and Terrone [26] are going to be used in the present particular setting.

For any $y_0 \in [-1, 1]$, let $\mathcal{A}_{i,y_0}$ be the set of measurable functions $\alpha : \mathbb{R}_+ \to A$ such that the function $y : \mathbb{R}_+ \to \mathbb{R}$ given by

$$y(t) \equiv y_0 + \int_0^t \alpha_i^\perp(s) ds \tag{2.1}$$

satisfies the state constraint $y(s) \in [-1, 1]$ for $s \in \mathbb{R}_+$. To summarize,

$$\mathcal{A}_{i,y_0} = \{\alpha \ : \ \mathbb{R}_+ \to A, \text{ measurable, such that } y(s) \in [-1, 1] \ \forall s \geq 0\}, \tag{2.2}$$

where $y(\cdot)$ is given by (2.1).

It is possible to define the effective Hamiltonians relative to the edges: let $H_i : [0, +\infty] \times [-1, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H_i(x, y, p) = \max_{a \in A} (-p \cdot a - \ell_i(x, y, a)).$$

Theorem 2.1 (Alvarez-Bardi [3]). *For any number $x_i \geq 0$ and for any $p_i \in \mathbb{R}$, there exists a unique real number $\overline{H}_i(x_i, p_i)$ such that the problem*

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \geq \overline{H}_i(x_i, p_i) \qquad y \in [-1, 1], \tag{2.3}$$

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y)) e_i^\perp \leq \overline{H}_i(x_i, p_i) \qquad y \in (-1, 1) \tag{2.4}$$

has a Lipschitz continuous viscosity solution $\chi_i : [-1,1] \to \mathbb{R}$.
Moreover, for any $y_0 \in [-1,1]$,

$$\overline{H}_i(x_i, p_i) = \lim_{\rho \to 0+} \rho w_{\rho, x_i, p_i}(y_0), \tag{2.5}$$

where

$$w_{\rho, x_i, p_i}(y_0) = \sup_{\mathcal{A}_{i,y_0}} \int_0^\infty e^{-\rho s} \left(-p_i \alpha_i(s) - \ell_i(x_i, y(s), \alpha(s))\right) ds,$$

and $\mathcal{A}_{i,y_0}$ and y are defined in (2.2) and (2.1). The limit in (2.5) is uniform w.r.t. y_0.

In [3], it is proved that for $y(s)$ and $\mathcal{A}_{i,y_0}$ as in (2.2), $\overline{H}_i(x_i, p_i)$ may be characterized with long time limits:

$$\overline{H}_i(x_i, p_i) = \sup_{\mathcal{A}_{i,y_0}} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left(-p_i \alpha_i(s) - \ell_i(x_i, y(s), \alpha(s))\right) ds, \tag{2.6}$$

for all $y_0 \in [-1,1]$.
In the same article [3], the following characterization of $\overline{H}_i(x_i, p_i)$ is given, as a rescaled limit of the value function of a finite horizon problem:

$$\overline{H}_i(x_i, p_i) = \lim_{t \to \infty} \sup_{\mathcal{A}_{i,y_0}} \frac{1}{t} \int_0^t \left(-p_i \alpha_i(s) - \ell_i(x_i, y(s), \alpha(s))\right) ds, \tag{2.7}$$

for all $y_0 \in [-1,1]$. For the latter, Alvarez and Bardi consider the value function

$$v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in \mathcal{A}_{i,y_0}} \left\{ \int_0^t \left(p_i \alpha_i(s) + \ell_i(x_i, y(s), \alpha(s))\right) ds \right\} \tag{2.8}$$

and prove that there exists a constant C independent of y_0 and t (but which may depend on x_i and p_i) such that

$$|v_i(x_i, p_i, y_0, t) + t\overline{H}_i(x_i, p_i)| \le C. \tag{2.9}$$

Limiting relaxed controls and limit control problem It is possible to construct an optimal control problem whose Hamiltonian is $\overline{H}_i$.
Let $([-1,1] \times A)^r$ be the set of Radon probability measures on $[-1,1] \times A$. For a function $f : \mathbb{R} \times [-1,1] \times A \to \mathbb{R}$, let f^r be defined by $f^r(x, \mu) = \int_{[-1,1]\times A} f(x, y, a) d\mu(y, a)$, for all $x \in \mathbb{R}$ and $\mu \in ([-1,1] \times A)^r$.
A sequence of Radon probability measures μ_n on $[-1,1] \times A$ is said to converge weak-$*$ to μ if for any continuous function ψ on $[-1,1] \times A$, $\lim_{n\to\infty} \int_{[-1,1]\times A} \psi(y, a) d\mu_n(y, a) = \int_{[-1,1]\times A} \psi(y, a) d\mu(y, a)$.
The Prokhorov distance $\pi(\cdot, \cdot)$ is defined for any pair (μ_1, μ_2) of probability measures on $[-1,1] \times A$ by

$$\pi(\mu_1, \mu_2) = \inf\{\varepsilon > 0 \ : \ \mu_1(Q) \le \mu_2(Q + \varepsilon B) + \varepsilon \text{ for any measurable } Q\},$$

where B is the unit ball in $\mathbb{R}^3$. It is well known that this distance has the following property: any sequence μ_n of Radon probability measures on $[-1,1] \times A$ converges weak-$*$ to μ if and only if $\lim_{n\to\infty} \pi(\mu_n, \mu) = 0$. The set $([-1,1] \times A)^r$ equipped with the distance π is a compact metric space.
If $\mathcal{P}$ is a subset of $([-1,1] \times A)^r$ and μ is an element of $([-1,1] \times A)^r$, then we define the distance

$\pi(\mu, \mathcal{P})$ as the infimum of $\{\pi(\mu,\nu), \nu \in \mathcal{P}\}$. For two subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ of $([-1,1]\times A)^r$, the Hausdorff distance $\pi_H(\mathcal{P}_1,\mathcal{P}_2)$ is defined as

$$\pi_H(\mathcal{P}_1,\mathcal{P}_2) = \max\left(\sup_{\mu\in\mathcal{P}_1}\pi(\mu,\mathcal{P}_2), \sup_{\mu\in\mathcal{P}_2}\pi(\mu,\mathcal{P}_1)\right).$$

For any control law $\alpha \in \mathcal{A}_{i,y_0}$, let $y(t)$ be given by (2.1). For $s>0$, the *occupational measure* μ_s generated by $(y(t),\alpha(t))$ is the Radon probability measure defined on the Borel σ-algebra of $\mathbb{R}\times A$ by

$$\mu_s = \frac{1}{s}\int_0^s \delta_{(y(t),\alpha(t))}dt$$

where $\delta_{(y(t),\alpha(t))}$ is the Dirac mass concentrated at $(y(t),\alpha(t))$.
For $s>0$, let $\mathcal{Z}(s;i,y_0)$ be the set of the occupational measures generated by the trajectories $(y(t),\alpha(t))$ up to time s where α belongs to $\mathcal{A}_{i,y_0}$ and $y(s)$ is given by (2.1). It has been proved in [18] that there exists a subset $\mathcal{Z}(i,y_0)$ of $([-1,1]\times A)^r$ such that

$$\lim_{s\to\infty}\pi_H(\mathcal{Z}(s;i,y_0),\mathcal{Z}(i,y_0)) = 0. \tag{2.10}$$

The set $\mathcal{Z}(i,y_0)$ is convex and compact in the weak-$*$ topology. Moreover, under Assumption 1.1, the set $\mathcal{Z}(i,y_0)$ does not depend on $y_0\in[-1,1]$. This set is called the *limit occupational measure set* and will be noted $\mathcal{Z}_i$.
It has been proved by Terrone[26] that $\mathcal{Z}_i$ coincides with the set of *limiting relaxed controls*, i.e. the set of the probability Radon measures μ on $[-1,1]\times A$ such that there exists a control law $\alpha\in\mathcal{A}_{i,y_0}$ and a sequence $t_n\to+\infty$ such that the corresponding sequence of occupational measures μ_{t_n} generated by (2.1) converges to μ weak-$*$.
Then using the results in [3], we see that

$$\mathcal{Z}_i \subset \left\{\mu\in([-1,1]\times A)^r, \int_{[-1,1]\times A} a_i^\perp d\mu(y,a) = 0\right\}.$$

It is proved in [3] that

$$\overline{H}_i(x_i,p_i) = \sup_{\mu\in\mathcal{Z}_i}\left(-p_i\int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(x_i,\mu)\right).$$

It is clear that $\overline{H}_i$ is continuous, convex with respect to the second variable, and that for all x_i, $\lim_{p_i\to\infty}\overline{H}_i(x_i,p_i)=+\infty$. The infinite horizon control problem associated with the Hamiltonian $\overline{H}_i$ is the minimization of

$$\overline{J}_i(x,\mu) = \int_0^\infty e^{-\lambda t}\ell_i^r(x(t),\mu_t)dt$$

for the system

$$\dot{x}(t) = \int_{[-1,1]\times A} a_i d\mu_t(y,a), \quad \mu_t\in\mathcal{Z}_i, \quad x(0)=x.$$

The affine-convex case It is proved in [3] that if A is convex and ℓ_i is convex with respect to its last two variables $(y_i^\perp, a)$, then $\overline{H}_i(x_i,p_i)$ is characterized by

$$\overline{H}_i(x_i,p_i) = \max_{a\in A, a_i^\perp=0, y\in[-1,1]}\Big(-p_i a_i - \ell_i(x_i,y,a)\Big). \tag{2.11}$$

If furthermore, ℓ_i does not depend on y, then

$$\overline{H}_i(x_i,p_i) = \max_{a\in A, a_i^\perp=0} \Big(-p_i a_i - \ell_i(x_i,a)\Big). \tag{2.12}$$

The case of an asymptotically stable optimal trajectory The characterization of $\overline{H}_i(x_i,p_i)$ by (2.11) can be obtained in other situations than the affine-convex case: here, following Example 7.4 in [7], we assume that for all x_i, p_i, there exists some $y_0 \in [-1,1]$ and some control $\alpha^* \in \mathcal{A}_{i,y_0}$ such that

$$\int_0^t \left(p_i\alpha_i^*(s) + \ell_i(x_i, y^*(s), \alpha^*(s))\right) ds = v_i(x_i,p_i,y_0,t), \quad \forall t>0,$$

where $\dot y^*(s) = \alpha_i^{*,\perp}(s)$ and $y^*(0)=y_0$, and that

$$\lim_{t\to\infty} \alpha^*(t) = a^* \quad \text{ and } \lim_{t\to\infty} y^*(t) = y^*.$$

Then $a_i^{*,\perp} = 0$ and $\lim_{t\to\infty} \frac{1}{t}\int_0^t \left(p_i\alpha_i^*(s) + \ell_i(x_i, y^*(s), \alpha^*(s))\right) ds = p_i a_i^* + \ell_i(x_i,y^*,a^*)$. This implies (2.11), see [3].

2.2 The minimal value of $\overline{H}_i(0,\cdot)$

Define Λ_i^0 by

$$\Lambda_i^0 = \min_{p\in\mathbb{R}} \overline{H}_i(0,p). \tag{2.13}$$

Since $\lim_{|p|\to\infty} \overline{H}_i(0,p) = +\infty$ and $\overline{H}_i(0,\cdot)$ is convex, the set $\operatorname{argmin} H_i(0,\cdot)$ is a nonempty bounded interval, see Figure 4 for an example: let $\underline{p}_i \geq \overline{p}_i$ be the endpoints of $\operatorname{argmin} H_i(0,\cdot)$:

$$\operatorname{argmin} H_i(0,\cdot) = \{p\in\mathbb{R},\ \overline{H}_i(0,p)=\Lambda_i^0\} = \left[\underline{p}_i, \overline{p}_i\right]. \tag{2.14}$$

Let us denote by $\mathcal{Z}_i^0$ and $\mathcal{Z}_i^+$ the following convex and compact subsets of $\mathcal{Z}_i$:

$$\mathcal{Z}_i^0 = \left\{\mu\in\mathcal{Z}_i \text{ s.t. } \int_{[-1,1]\times A} a_i d\mu(y,a) = 0\right\}, \tag{2.15}$$

$$\mathcal{Z}_i^+ = \left\{\mu\in\mathcal{Z}_i \text{ s.t. } \int_{[-1,1]\times A} a_i d\mu(y,a) \geq 0\right\}. \tag{2.16}$$

These sets are non empty from Assumption 1.1.

Lemma 2.1. *1. $p_i^0 \in \operatorname{argmin} H_i(0,\cdot)$ if and only if there exists $\mu^*\in\mathcal{Z}_i^0$ such that*

$$\overline{H}_i(0,p_i^0) = -\ell_i^r(0,\mu^*)$$

2.

$$\Lambda_i^0 = -\min_{\mu\in\mathcal{Z}_i^0} \ell_i^r(0,\mu) \tag{2.17}$$

3. If $p\geq \underline{p}_i$, then

$$\max_{\mu\in\mathcal{Z}_i^+} \int_{[-1,1]\times A} (-pa_i - \ell_i(0,y,a))d\mu(y,a) = \Lambda_i^0.$$

Proof. The Hamiltonian $\overline{H}_i(0,\cdot)$ reaches its minimum at p_i^0 if and only if $0 \in \partial \overline{H}_i(0,p_i^0)$. The subdifferential of $\overline{H}_i(0,\cdot)$ at p_i^0 is characterized by

$$\partial \overline{H}_i(0,p_i^0) = \overline{\text{co}} \left\{ -\int_{[-1,1]\times A} a_i d\mu(y,a); \mu \in \mathcal{Z}_i \text{ s.t. } \overline{H}_i(0,p_i^0) = -p_i^0 \int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(0,\mu) \right\},$$

see [27]. But the set

$$\left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \overline{H}_i(0,p_i^0) = -p_i^0 \int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(0,\mu) \right\}$$

is compact and convex. Hence,

$$\partial \overline{H}_i(0,p_i^0) = \left\{ -\int_{[-1,1]\times A} a_i d\mu(y,a); \mu \in \mathcal{Z}_i \text{ s.t. } \overline{H}_i(0,p_i^0) = -p_i^0 \int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(0,\mu) \right\}.$$

Therefore, $0 \in \partial \overline{H}_i(0,p_i^0)$ if and only if there exists $\mu^* \in \mathcal{Z}_i$ such that $\int_{[-1,1]\times A} a_i d\mu^*(y,a) = 0$ and $\overline{H}_i(0,p_i^0) = -\ell_i^r(0,\mu^*)$. We have proved point 1.
Point 2 is a direct consequence of point 1.
If $p \geq \underline{p}_i$, then

$$\max_{\mu \in \mathcal{Z}_i^+} \left(-p \int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(0,\mu) \right) \leq \max_{\mu \in \mathcal{Z}_i^+} \left(-\underline{p}_i \int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(0,\mu) \right) = \overline{H}_i(0,\underline{p}_i)$$

where the last identity comes from point 1.
On the other hand,

$$\max_{\mu \in \mathcal{Z}_i^+} \left(-p \int_{[-1,1]\times A} a_i d\mu(y,a) - \ell_i^r(0,\mu) \right) \geq -\min_{\mu \in \mathcal{Z}_i^0} \ell_i^r(0,\mu).$$

Point 3 is obtained by combining the two previous observations and point 2. □

Remark 2.1. *From Assumption 1.3 and Lemma 2.1, we see that Λ_i^0 is non positive.*

2.3 A further assumption on $\ell_i(0,\cdot,\cdot)$

Assumption 2.1 below will turn useful for studying the asymptotics of u_ε near the junction; roughly speaking, it says that for any real number $p_i < \underline{p}_i$ (which implies $\overline{H}_i(0,p_i) > \Lambda_i^0$), for all t, it is possible to construct an admissible control law $\tilde{\alpha}$ whose related cost remains at a given distance of the optimal value $v_i(0,p_i,y_0,t)$, and for which the i coordinate, namely $s \in [0,t] \mapsto \int_0^s \tilde{\alpha}_i(\theta)d\theta$ remains bounded from below by a fixed constant. Assumption 2.1 will be used in § 5 for constructing the correctors near the junction.
Proposition 2.1 and 2.2 below supply examples when Assumption 2.1 holds.

Assumption 2.1. *1. For any real number p_i such that $p_i < \underline{p}_i$, there exist two constants $L_i \geq 0$ and $C_i > 0$ such that for all $y_0 \in [-1,1]$, for all $t > 0$, there exists a control law $\tilde{\alpha} \in \mathcal{A}_{i,y_0}$ such that*

$$\int_0^s \tilde{\alpha}_i(\tau)d\tau \geq -L_i, \quad \forall 0 \leq s \leq t, \tag{2.18}$$

$$\int_0^t \left(p_i\tilde{\alpha}_i(s) + \ell_i(0,\tilde{y}(s),\tilde{\alpha}(s))\right) ds \leq v_i(0,p_i,y_0,t) + C_i, \tag{2.19}$$

where $\tilde{y}(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s)ds$.

2. *For* p_i *such that* $\underline{p}_i \le p_i \le \overline{p}_i$, *(hence* $\overline{H}_i(0,p_i) = \Lambda_i^0$*), there exist two constants* $L_i \ge 0$ *and* $C_i > 0$ *such that for all* $y_0 \in [-1,1]$, *for all* $t > 0$, *there exists a control law* $\tilde{\alpha} \in \mathcal{A}_{i,y_0}$ *such that*

$$-L_i \le \int_0^s \tilde{\alpha}_i(\tau)d\tau \le L_i, \quad \forall 0 \le s \le t, \tag{2.20}$$

$$\int_0^t \ell_i(0, \tilde{y}(s), \tilde{\alpha}(s))ds \le v_i(0, p_i, y_0, t) + C_i, \tag{2.21}$$

where $\tilde{y}(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s)ds$.

Remark 2.2. *Note that from (2.9), (2.19) is equivalent to*

$$\int_0^t (p_i\tilde{\alpha}_i(s) + \ell_i(0, \tilde{y}(s), \tilde{\alpha}(s)))\, ds \le -t\overline{H}_i(0,p_i) + C_{i,2}, \tag{2.22}$$

for a constant $C_{i,2}$, *and that (2.20)(2.21) imply (2.22).*

Proposition 2.1. *Under Assumptions 1.1, 1.2 and 1.3, if the set* A *is convex and the function* $\ell_i(0,\cdot,\cdot)$ *is convex on* $[-1,1] \times A$, *then Assumption 2.1 holds.*

Proof. The proof is given in the appendix. □

Remark 2.3. *The conclusion of Proposition 2.1 holds when there is an asymptotically stable optimal trajectory, see the paragraph at the end of § 2.1.*

Proposition 2.2. *Under Assumptions 1.1, 1.2 and 1.3, if there exist two positive constants* $\underline{c}_i \le \overline{c}_i$ *and an exponent* $\nu_i > 1$ *such that for all* $y \in [-1,1]$ *and* $a \in A$,

$$-\Lambda_i^0 + \underline{c}_i|a|^{\nu_i} \le \ell_i(0,y,a) \le -\Lambda_i^0 + \overline{c}_i|a|^{\nu_i}, \tag{2.23}$$

where Λ_i^0 *is defined in (2.13), then Assumption 2.1 holds.*

Proof. The proof is given in the appendix. □

3 A scalar quantity that will appear in the effective equation at the junction

In what follows, we will make a blow-up near the junction point O: this leads us to extend the function ℓ_0 to the whole domain $\overline{\Omega}$ by setting for any $a \in A$

$$\ell_0(z,a) = \ell_i(0, z_i^\perp, a), \quad \text{if } z_i \ge r_0,\ |z_i^\perp| \le 1. \tag{3.1}$$

3.1 Ergodic constants for state constraint problems in bounded subdomains

For any real number R such that $R > r_0$, let the bounded and connected open set Ω^R be defined by

$$\Omega^R = \Omega \cap W_R, \tag{3.2}$$

where W_R is given by (1.1). Consider the infinite horizon control problem

$$w_\rho^R(z_0) = \inf_{\mathcal{A}_{z_0}^R} \int_0^\infty e^{-\rho s}\ell_0(z(s), \alpha(s))ds, \tag{3.3}$$

where

$$\mathcal{A}_{z_0}^R = \left\{ \alpha \,:\, \mathbb{R}_+ \to A, \text{ measurable, such that } z(s) \equiv z_0 + \int_0^s \alpha(\theta) d\theta \in \overline{\Omega}^R, \forall s \geq 0 \right\}. \quad (3.4)$$

It is easy to prove that w_ρ^R is the unique viscosity solution of the following problem:

$$\rho w_\rho^R + H_0(\cdot, Dw_\rho^R) \geq 0 \qquad \text{in } \overline{\Omega}^R, \quad (3.5)$$

$$\rho w_\rho^R + H_0(\cdot, Dw_\rho^R) \leq 0 \qquad \text{in } \Omega^R, \quad (3.6)$$

where for $z \in \overline{\Omega}^R$, $p \in \mathbb{R}^2$, the Hamiltonian $H_0(z,p)$ is

$$H_0(z,p) = \max_{a \in A} \Big(-p \cdot a - \ell_0(z,a) \Big). \quad (3.7)$$

Using the hypotheses on controllability, continuity and boundedness of the data, it is now standard (see [24]) to obtain the existence of a unique ergodic constant E^R and of a bounded Lipschitz corrector w^R such that

$$H_0(\cdot, Dw^R) \geq E^R \qquad \text{in } \overline{\Omega}^R, \quad (3.8)$$

$$H_0(\cdot, Dw^R) \leq E^R \qquad \text{in } \Omega^R, \quad (3.9)$$

where (up to a subsequence) $w^R(z) = \lim_{\rho \to 0} w_\rho^R(z) - < w_\rho^R >$ and $E^R = -\lim_{\rho \to 0} \rho w_\rho^R(z)$.
The ergodic constant E^R is bounded uniformly w.r.t. R and there exists a constant C_L independent of R such that $|Dw^R(z)| \leq C_L$. Furthermore, if $R \leq R'$, the inclusion $\mathcal{A}_z^R \subset \mathcal{A}_z^{R'}$ yields that $w_\rho^R(z) \geq w_\rho^{R'}(z)$. This implies that if $R \leq R'$, then $E^R \leq E^{R'}$.

3.2 Passing to the limit as $R \to +\infty$: the constant E

The boundedness and the monotony of the application $R \mapsto E^R$ make it possible to pass to the limit and define a limit ergodic constant

$$E \equiv \lim_{R \to \infty} E^R. \quad (3.10)$$

The constant E will play an important role in the effective equation at the junction.

Example 3.1. *It is easy to see that if there exists $z_0 \in \tilde{K}_0$ such that $\ell_0(z_0, 0) = 0$, then $E = 0$.*

Remark 3.1. *In what follows, we shall use the fact that, for any $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that $\forall r > R(\varepsilon)$, w^r is a viscosity supersolution of*

$$H_0(\cdot, Dw^r) \geq E - \varepsilon, \quad \text{in } \overline{\Omega}^r. \quad (3.11)$$

Remark 3.2. *Using the uniform coercivity of H_0, there exists q such that $H_0(z,p) \geq E + 2$ for any $z \in \overline{\Omega}$, $p \in \mathbb{R}^2$ with $|p| \geq q$.*
Let R be any fixed real number such that $R > r_0$. For any $M \geq q$, consider the continuous extension $\tilde{w}^R$ of w^R, defined for $z = z_i e_i + z_i^\perp e_i^\perp$, $z_i \in (R, +\infty)$, $z_i^\perp \in [-1,1]$ by

$$\tilde{w}^R(z) := w^R(z^R) + M(z - z^R), \quad \text{with } z^R = Re_i + z_i^\perp e_i^\perp. \quad (3.12)$$

It is clear that $\tilde{w}^R$ is a continuous extension of w^R to $\overline{\Omega}$.
Moreover, for $R > R(\varepsilon)$ as in Remark 3.1, it is easy to see that $\tilde{w}^R$ is a viscosity supersolution of

$$H_0(\cdot, D\tilde{w}^R) \geq E - \varepsilon \quad \text{in } \overline{\Omega}. \quad (3.13)$$

3.3 Estimates for some trajectories which start in $\widetilde{K}_0$ and return there after some time

For $z_0 \in \overline{\Omega}$, let $\mathcal{A}_{z_0}$ be the set

$$\mathcal{A}_{z_0} = \left\{ \alpha \,:\, \mathbb{R}_+ \to A, \text{ measurable, such that } \forall s \geq 0,\ z(s) \equiv z_0 + \int_0^s \alpha(\theta) d\theta \in \overline{\Omega} \right\}.$$

Lemma 3.1. *There exists a constant C such that for all $z_0 \in \widetilde{K}_0$, (see (1.3) for the definition of $\widetilde{K}_0$), for all $\alpha \in \mathcal{A}_{z_0}$ and $T_0 > 0$ such that $z(T_0) \in \widetilde{K}_0$, (recall that $z(s) \equiv z_0 + \int_0^s \alpha(\theta)$),*

$$\int_0^{T_0} \ell_0(z(s), \alpha(s)) ds \geq -ET_0 - C.$$

Proof. There exists a time $\tau \geq 0$ (which can be bounded by a constant depending only on the controllability and bounds in our hypotheses (on ℓ and A)) and a control law $\tilde{\alpha}(\cdot)$ such that $z(T_0) + \int_0^t \tilde{\alpha}(s) ds \in \widetilde{K}_0$ for any $t \in [0, \tau]$ and $z(T_0) + \int_0^\tau \tilde{\alpha}(s) ds = z_0$. The following control law

$$\alpha^*(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, T_0], \\ \tilde{\alpha}(t - T_0) & \text{if } t \in (T_0, T_0 + \tau], \end{cases}$$

can be extended by periodicity to $\mathbb{R}^+$ (the period is $T_0 + \tau$) to yield an admissible periodic trajectory $z^*(t) \equiv z_0 + \int_0^t \alpha^*(s) ds$. Then,

$$\int_0^\infty e^{-\rho s} \ell_0(z^*(s), \alpha^*(s)) ds = \sum_{k=0}^\infty \int_{k(T_0+\tau)}^{(k+1)(T_0+\tau)} e^{-\rho s} \ell_0(z^*(s), \alpha^*(s)) ds,$$

and

$$\sum_{k=0}^\infty e^{-k\rho(T_0+\tau)} \left(\int_0^{T_0+\tau} e^{-\rho s} \ell_0(z^*(s), \alpha^*(s)) ds \right) = \frac{1}{1 - e^{-\rho(T_0+\tau)}} \int_0^{T_0+\tau} e^{-\rho s} \ell_0(z^*(s), \alpha^*(s)) ds.$$

This implies that

$$\lim_{\rho \to 0} \rho \int_0^\infty e^{-\rho s} \ell_0(z^*(s), \alpha^*(s)) ds = \frac{1}{T_0 + \tau} \int_0^{T_0+\tau} \ell_0(z^*(s), \alpha^*(s)) ds. \tag{3.14}$$

Moreover, it is possible to choose R large enough such that $z^*(t) \in \Omega^R$ for all $t \in [0, T_0 + \tau]$. Therefore, up to a subsequence,

$$\lim_{\rho \to 0} \rho \int_0^\infty e^{-\rho s} \left(\ell_0(z^*(s), \alpha^*(s)) \right) ds \geq \lim_{\rho \to 0} \rho w_\rho^R(z_0) = E^R \geq -E,$$

and this implies thanks to (3.14) that

$$\frac{1}{T_0 + \tau} \int_0^{T_0+\tau} \ell_0(z^*(s), \alpha^*(s)) ds \geq -E.$$

Therefore,

$$\int_0^{T_0} \ell_0(z(s), \alpha(s)) ds \geq -E(T_0 + \tau) - \int_{T_0}^{T_0+\tau} \ell_0(z^*(s), \alpha^*(s)) ds \geq -ET_0 - C.$$

□

3.4 Generalization of the previous results

Let R_0 be a fixed real number such that $R_0 > r_0$. It will prove useful to define the following open sets, see Figure 2:

$$K_i = Z_i \cap \{x : x_i > R_0\}, \quad i = 1, \dots, N, \tag{3.15}$$

$$\widetilde{K}_i = Z_i \cap \{x : r_0 < x_i < R_0\}, \quad i = 1, \dots, N, \tag{3.16}$$

$$\omega = \Omega \cap \{x : x_i < R_0, \ \forall i = 1, \dots, N\}, \tag{3.17}$$

with Z_i defined in (1.2). Note that $\overline{\omega} = \cup_{i=0}^{N} \overline{\widetilde{K}_i}$ and that $\overline{\Omega} = \overline{\omega} \cup \bigcup_{i=1}^{N} \overline{K}_i$.

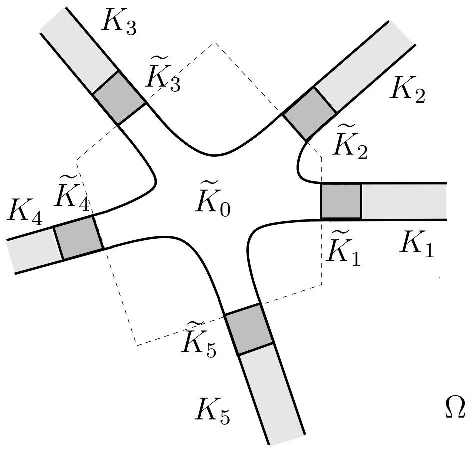

Figure 2: The set Ω is partitioned into different zones

Consider a subdomain $\tilde{\Omega}$ of Ω obtained as follows: call $\mathcal{I}$ a subset of $\{1, \dots, N\}$ and define $\tilde{\Omega}$ by

$$\overline{\tilde{\Omega}} = \overline{\widetilde{K}_0} \cup \bigcup_{i \in \mathcal{I}} \overline{\widetilde{K}_i} \cup \bigcup_{i \in \mathcal{I}} \overline{K_i},$$

see Figure 3.

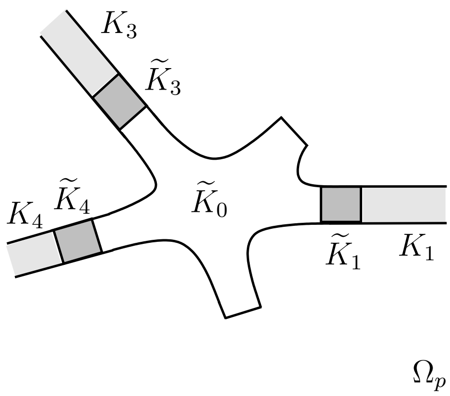

Figure 3: The set $\tilde{\Omega}$ in the case when $\mathcal{I} = \{1, 3, 4\}$

Let $\tilde{\Omega}^R = \tilde{\Omega} \cap W^R$. As in § 3.1, it is possible to obtain the existence of a unique ergodic constant $E^R_{\tilde\Omega}$ and of a bounded Lipschitz corrector $w^R_{\tilde\Omega}$ such that

$$\begin{aligned} H_0(\cdot, Dw^R_{\tilde\Omega}) &\ge E^R_{\tilde\Omega} \qquad \text{in } \overline{\tilde\Omega^R},\\ H_0(\cdot, Dw^R_{\tilde\Omega}) &\le E^R_{\tilde\Omega} \qquad \text{in } \tilde\Omega^R, \end{aligned}$$

and $E^R_{\tilde\Omega} \le E^R \le E$ since $\tilde\Omega^R \subset \Omega^R$.

Remark 3.3. *Remarks 3.1 and 3.2 hold with the obvious changes consisting of replacing Ω, (resp. Ω^r, Ω^R, w^R, $\tilde w^R$) with $\tilde\Omega$, (resp. $\tilde\Omega^r$, $\tilde\Omega^R$, $w^R_{\tilde\Omega}$, $\tilde w^R_{\tilde\Omega}$).*

For $z_0 \in \overline{\tilde\Omega}$, let $\mathcal{A}^{\tilde\Omega}_{z_0}$ be the set

$$\mathcal{A}^{\tilde\Omega}_{z_0} = \left\{ \alpha \,:\, \mathbb{R}_+ \to A, \text{ measurable, such that } \forall s \ge 0,\ z(s) \equiv z_0 + \int_0^s \alpha(\theta)d\theta \in \overline{\tilde\Omega} \right\}.$$

The following lemma is proved exactly as Lemma 3.1:

Lemma 3.2. *There exists a constant C such that for all $z_0 \in \widetilde{K}_0$, (see (1.3) for the definition of $\widetilde{K}_0$), for all $\alpha \in \mathcal{A}^{\tilde\Omega}_{z_0}$ and $T_0 > 0$ such that $z(T_0) \in \widetilde{K}_0$, (recall that $z(s) \equiv z_0 + \int_0^s \alpha(\theta)d\theta$),*

$$\int_0^{T_0} \ell_0(z(s), \alpha(s))ds \ge -ET_0 - C.$$

4 The main result: the effective problem and the convergence theorem

The aim is first to introduce an effective Hamiltonian at the junction, noted $\overline{H}(O, \cdot)$ below and to define the effective problem. Then we will briefly summarize the results of Imbert-Monneau [20] on Hamilton-Jacobi equations on $\mathcal{G}$. Finally we will state the main convergence result.

4.1 The effective Hamiltonian at the junction

Let $\overline{H}_i^+(x_i, p_i)$ be defined by

$$\overline{H}_i^+(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left(-p_i \int_{[-1,1]\times A} a_i d\mu(y, a) - \ell_i^r(x, \mu) \right) \tag{4.1}$$

where $\mathcal{Z}_i^+$ is defined by (2.16).

Remark 4.1. *In the affine-convex case, H_i^+ is characterized by*

$$\overline{H}_i^+(x_i, p_i) = \max_{a \in A, a_i \ge 0, a_i^\perp = 0, y_i^\perp \in [-1,1]} \Big(-p_i a_i - \ell_i(x_i, y_i^\perp, a) \Big). \tag{4.2}$$

Let the function $\overline{H}(O, \cdot) : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\overline{H}(O, q) = \max_{i=1,\dots,N} \overline{H}_i^+(0, q_i), \quad \forall q \in \mathbb{R}^N, \tag{4.3}$$

and the set of indices $\mathcal{I}(q)$ be defined by

$$\mathcal{I}(q) = \{ i : 1 \le i \le N,\ \overline{H}_i^+(0, q_i) = \overline{H}(O, q) \}. \tag{4.4}$$

We also define what may be called the tangential Hamiltonian at the junction (see [2]) by

$$\Lambda = \max_{i=1,\dots,N} \Lambda_i^0, \tag{4.5}$$

where $\Lambda_i^0 = \min_{p\in\mathbb{R}} \overline{H}_i(0,p)$.

Remark 4.2. *From Lemma 2.1, we see that*

$$\overline{H}_i^+(0,p_i) = \begin{cases} \overline{H}_i(0,p_i) > \Lambda_i^0 & \text{if } p_i < \underline{p}_i, \\ \Lambda_i^0 & \text{if } p_i \geq \underline{p}_i, \end{cases} \tag{4.6}$$

then, it is clear that $\forall q \in \mathbb{R}^N$,

$$\overline{H}(O,q) \geq \Lambda. \tag{4.7}$$

In Figure 4, we give an example for the graphs of $p \mapsto \overline{H}_i(0,p)$ *and of* $p \mapsto \overline{H}_i^+(0,p)$. *The constant* Λ_i^0 *is the minimal value of both* $\overline{H}_i(0,\cdot)$ *and* $\overline{H}_i^+(0,\cdot)$.

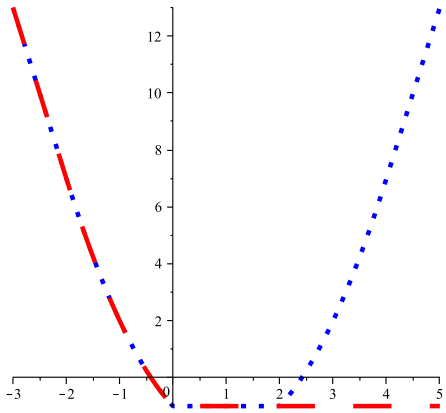

Figure 4: The graphs of the Hamiltonian $p \mapsto \overline{H}_i(0,p)$ and of $p \mapsto \overline{H}_i^+(0,p)$ coincide for $p \leq \bar{p}_i$, and $\overline{H}_i^+(0,p) = \Lambda_i^0$ for $p \geq \underline{p}_i$. In the example, $\underline{p}_i = 0$ and $\bar{p}_i = 2$.

Lemma 4.1. *Under Assumptions 1.1- 1.3 and 2.1,* Λ *defined in (4.5) and* E *defined in (3.10) satisfy*

$$\Lambda \leq E. \tag{4.8}$$

Proof. For any $i = 1,\dots,N$, choose $p_i \in \operatorname{argmin} \overline{H}_i(0,\cdot)$, and L_i as in Assumption 2.1. Take $z_{0,i} > 2r_0 + L_i$, $z_0 = z_{0,i}e_i + z_{0,i}^\perp e_i^\perp \in \overline{\Omega}$ and any $R > z_i + L_i$. From Assumption 2.1 and Remark 2.2, we know that for all $t > 0$, there exists a control law $\alpha \in \mathcal{A}_{z_0}$ such that for all $s \in [0,t]$, $z(s) = z_0 + \int_0^s \alpha(\theta)d\theta$ belongs to the set $\{x : \ x_i \in [r_0, z_{0,i} + L_i], x_i^\perp \in [-1,1]\} \subset \overline{\Omega}^R$ and

$$\int_0^t \ell_0(z(s),\alpha(s))ds \leq -t\overline{H}_i(0,p_i) + C, \tag{4.9}$$

for a constant C independent of t.
From Assumption 1.1, there exists a finite time $\tau > 0$ (which is bounded uniformly w.r.t. t) and $a \in A$ such that $z(t) + a\tau = z_0$. The following control law

$$\alpha^*(s) = \begin{cases} \alpha(s) & \text{if } s \in [0,t], \\ a & \text{if } s \in (t, t+\tau], \end{cases}$$

can be extended by periodicity to $\mathbb{R}^+$ (the period is $t+\tau$) to yield a periodic trajectory $z^*(s) \equiv z_0 + \int_0^s \alpha^*(\theta)d\theta$, which stays in $\overline{\Omega}^R$. The arguments contained in the proof of Lemma 3.1 can then be repeated to obtain that

$$\frac{1}{t+\tau}\int_0^{t+\tau} \ell_0(z^*(s),\alpha^*(s))ds \geq -E,$$

and that

$$\int_0^t \ell_0(z(s),\alpha(s))ds \geq -Et - \tilde{C}$$

for another constant $\tilde{C}$. Combining this with (4.9) yields that $-Et - \tilde{C} \leq -t\overline{H}_i(0,p_i) + C = -t\Lambda_i^0 + C$. Since t can be chosen as large as desired, we get that $\Lambda_i^0 \leq E$, and since the argument above can be repeated for all $i = 1,\ldots,N$, we obtain (4.8). $\square$

4.2 The effective problem on $\mathcal{G}$

Here we introduce the effective problem on the network $\mathcal{G}$ and define its viscosity solution. We also recall some of the results obtained by Imbert and Monneau [20], in a presentation adapted to the context.
The effective Hamilton-Jacobi equation on $\mathcal{G}$ is:

$$\lambda u(x) + \overline{H}_i(x_i, Du(x)) = 0 \quad x = x_i e_i \in G_i, \tag{4.10}$$
$$\lambda u(O) + \max(E, \overline{H}(O, Du(O))) = 0 \quad x = O, \tag{4.11}$$

where $\overline{H}(O,\cdot)$ is defined in (4.3), E is defined in (3.10) and

$$Du(x) = \begin{cases} \frac{du}{dx_i}(x) & \text{if } x \in G_i, \\ (\frac{du}{dx_1}(O),\ldots,\frac{du}{dx_N}(O)) & \text{if } x = O. \end{cases} \tag{4.12}$$

4.2.1 Test functions

For the definition of viscosity solutions on the irregular set $\mathcal{G}$, it is necessary to first define a class of the admissible test functions

Definition 4.1. *A function $\varphi : \mathcal{G} \to \mathbb{R}$ is an admissible test function if*

- *φ is continuous in $\mathcal{G}$ and $\mathcal{C}^1$ in $\mathcal{G}\setminus\{O\}$*
- *for any j, $j = 1,\ldots,N$, $\varphi|_{\bar{G}_j} \in \mathcal{C}^1(\bar{G}_j)$.*

The set of admissible test function is noted $\mathcal{R}(\mathcal{G})$. If $\varphi \in \mathcal{R}(\mathcal{G})$ and $\zeta \in \mathbb{R}$, let $D\varphi(x,\zeta e_i)$ be defined by $D\varphi(x,\zeta e_i) = \zeta\frac{d\varphi}{dx_i}(x)$ if $x \in G_i\setminus\{O\}$ and $D\varphi(O,\zeta e_i) = \zeta\lim_{h\to 0+}\frac{d\varphi}{dx_i}(he_i)$.

Property 4.1. *If $\varphi = g\circ\psi$ with $g \in \mathcal{C}^1$ and $\psi \in \mathcal{R}(\mathcal{G})$, then $\varphi \in \mathcal{R}(\mathcal{G})$ and*

$$D\varphi(O,\zeta) = g'(\psi(O))D\psi(O,\zeta).$$

4.2.2 Definition of viscosity solutions

Definition 4.2. • *An upper semi-continuous function $u : \mathcal{G} \to \mathbb{R}$ is a viscosity subsolution of (4.10-4.11) in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x, then*

$$\begin{aligned} &\lambda u(x) + \overline{H}_i(x, \tfrac{d\varphi}{dx_i}(x)) \le 0 && \text{if } x \in G_i, \\ &\lambda u(O) + \max(E, \overline{H}(O, \tfrac{d\varphi}{dx_1}(O), \dots, \tfrac{d\varphi}{dx_N}(O))) \le 0. \end{aligned} \tag{4.13}$$

- *A lower semi-continuous function $u : \mathcal{G} \to \mathbb{R}$ is a viscosity supersolution of (4.10-4.11) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x, then*

$$\begin{aligned} &\lambda u(x) + \overline{H}_i(x, \tfrac{d\varphi}{dx_i}(x)) \ge 0 && \text{if } x \in G_i, \\ &\lambda u(O) + \max(E, \overline{H}(O, \tfrac{d\varphi}{dx_1}(O), \dots, \tfrac{d\varphi}{dx_N}(O))) \ge 0. \end{aligned} \tag{4.14}$$

- *We say that u is a viscosity solution of (4.10-4.11) in $\mathcal{G}$ if it is both a viscosity subsolution and supersolution of (4.10-4.11).*

4.2.3 Comparison principle

The following result is a consequence of the general comparison results proved in [20]:

Theorem 4.1. *Under the Assumptions 1.1- 1.3, for all viscosity subsolution u and all supersolution v of (4.10-4.11) satisfying*

$$u(x) \le C(1 + |x|) \quad \text{and} \quad v(x) \ge -C(1 + |x|)$$

for some positive number C, we have

$$u \le v \qquad \text{in } \mathcal{G}. \tag{4.15}$$

4.3 The convergence result

We are ready to the state the main result of the paper:

Theorem 4.2 (Convergence)**.** *Under Assumptions 1.1-1.3 and 2.1, as $\varepsilon \to 0+$, the functions u_ε converge locally uniformly to the unique viscosity solution u of (4.10)-(4.11).*

The proof of this Theorem is given in Section 6. It is based on the perturbed test-function method of Evans. The perturbed test-functions involve suitable correctors which are constructed in Section 5.

5 Correctors at the junction when $\overline{H}(O, p) > E$

The aim here is to construct correctors (in unbounded domains) which will be used in § 6 in the perturbed test-function method of Evans, [17]. The strategy for constructing the correctors is reminiscent of the one used by Ishii in [22], although a new difficulty arises from the unboundedness of the domains in which the correctors will be defined.

Remark 5.1. *The question of the correctors in unbounded domains has very recently been addressed by P-L. Lions in his lectures at Collège de France, [23], precisely in january and february 2014 (therefore after the completion of the present work): the lectures dealt with recent and still unpublished results obtained in collaboration with T. Souganidis on the asymptotic behavior of solutions of Hamilton-Jacobi equations in a periodic framework with some localized defects. In this context, P-L. Lions addressed similar phenomena as those mentioned below, namely the possible nonexistence of correctors.*

Let $p \in \mathbb{R}^N$ be such that $\overline{H}(O,p) > \Lambda$. Then from (4.6), we see that for any index $i \in \mathcal{I}(p)$, $p_i < \underline{p}_i$ and $\overline{H}(O,p) = \overline{H}_i^+(0,p_i) = \overline{H}_i(0,p_i)$.
Take $R_0 > r_0$ as in § 3.4 and let ψ_p be a smooth function on $\overline{\Omega}$ such that

$$\begin{array}{rcll} \psi_p(z) & = & p_i z_i & \forall z \in \overline{K_i}, \\ \psi_p(z) & = & p_i \Psi(z_i) & \forall z \in \widetilde{K}_i, \\ \psi_p(z) & = & 0 & \forall z \in \widetilde{K}_0, \end{array} \tag{5.1}$$

where Ψ is a smooth non decreasing function defined on $[r_0, R_0]$ such that $\Psi = 0$ in a neighborhood of r_0, $\Psi(r) = r$ in a neighborhood of R_0, and $\Psi(r) \le r$ for all $r \in [r_0, R_0]$.
Let the open connected set Ω_p be defined by

$$\overline{\Omega}_p = \overline{\widetilde{K}_0} \cup \bigcup_{i \in \mathcal{I}(p)} \overline{\widetilde{K}_i} \cup \bigcup_{i \in \mathcal{I}(p)} \overline{K_i}. \tag{5.2}$$

where $\widetilde{K}_0$, $\widetilde{K}_i$ and K_i are defined respectively in (1.3), (3.16) and (3.15), see Figure 5. Note that $\overline{\Omega}_p$ corresponds to a set $\tilde{\Omega}$ defined in § 3.4.

Below, a corrector associated to p will be defined in $\overline{\Omega}_p$. Its existence will be stated in Theorem 5.1. Lemmas 5.1, 5.2 and 5.3 below are the main steps for proving Theorem 5.1.

Remark 5.2. *The idea of truncating the domain and use Ω_p instead of Ω comes from the fact that we need a bounded corrector, and constructing a bounded corrector does not seem possible in the full domain $\overline{\Omega}$. Since the correctors are defined in subdomains, the method of the perturbed test functions of Evans will have to be suitably modified, see § 6.*

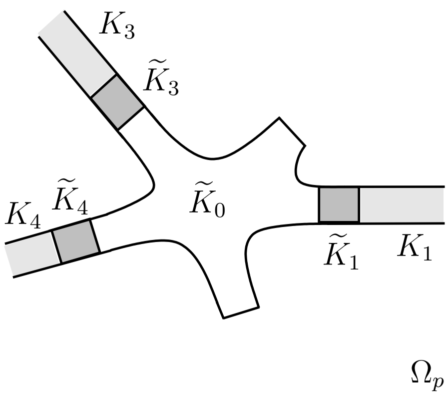

Figure 5: The set Ω_p in the case when $\mathcal{I}(p) = \{1, 3, 4\}$

For $z_0 \in \overline{\Omega}_p$ and $t \ge 0$, let $u(z_0, t)$ be defined by

$$u(z_0, t) = \inf_{\alpha \in \mathcal{A}_y^p} J(z_0, t, \alpha), \tag{5.3}$$

where

$$J(z_0, t, \alpha) = \int_0^t \left(D\psi_p(z(s)) \cdot \alpha(s) + \ell_0(z(s), \alpha(s)) \right) ds, \tag{5.4}$$

and

$$\dot{z}(s) = \alpha(s), \quad s > 0, \qquad z(0) = z_0,$$
$$\mathcal{A}^p_{z_0} = \{\alpha : \alpha(s) \in A \text{ for a.a. } s \; ; \; z(s) \in \overline{\Omega}_p \text{ for all } s\}.$$

It is well known, see [9], that u is a Lipschitz continuous viscosity solution of

$$\frac{\partial u}{\partial t}(z,t) + H_0(z, D\psi_p + Du) \geq 0 \qquad \text{in } \overline{\Omega}_p \times \mathbb{R}_+, \tag{5.5}$$
$$\frac{\partial u}{\partial t}(z,t) + H_0(z, D\psi_p + Du) \leq 0 \qquad \text{in } \Omega_p \times \mathbb{R}_+, \tag{5.6}$$

where

$$H_0(z, q) = \max_{a \in A} \left(-q \cdot a - \ell_0(z, a) \right). \tag{5.7}$$

Lemma 5.1. *Let $p \in \mathbb{R}^N$ be such that $\overline{H}(O, p) > \Lambda$. Under Assumptions 1.1-1.3 and 2.1, there exists a constant C such that for all $z \in \overline{\Omega}_p$ and $t \geq 0$,*

$$u(z, t) \leq -t\overline{H}(O, p) + C. \tag{5.8}$$

Proof. Consider $z_0 \in \overline{\Omega}_p$. We make out several cases:

Case 1 There exists i in $\mathcal{I}(p)$ such that $z_0 \in K_i$ and $z_{0,i} > L_i + R_0$ where L_i is the constant (possibly depending on p_i) appearing in Assumption 2.1. Consider the control law $\tilde{\alpha}$ described in Assumption 2.1 with $y_0 = z_{0,i}^{\perp}$. This control law is such that the corresponding trajectory $\tilde{z}(s) = z_0 + \int_0^s \tilde{\alpha}(\theta) d\theta$ remains in $\overline{K}_i$ for all $s \in [0, t]$. Moreover,

$$J(z_0, t, \tilde{\alpha}) = \int_0^t p_i \tilde{\alpha}_i(s) + \ell_i(0, \tilde{z}_i^{\perp}(s), \tilde{\alpha}(s)) ds \leq -t\overline{H}(O, p) + C_{i,2},$$

from (2.22). Hence

$$u(z_0, t) \leq -t\overline{H}(O, p) + C_{i,2}. \tag{5.9}$$

Case 2 In the opposite case, there exists a constant $\tilde{L}$ (depending on p) such that $|z_0| \leq \tilde{L}$. Let i_0 be an index $i_0 \in \mathcal{I}(p)$ minimizing distance$(z_0 - L_j e_j, \overline{K}_j)$. From the controllability assumption 1.1, it is possible to find $T > 0$ such that for all z_0 in Case 2, there exists a control law $\bar{\alpha}$ such that $\bar{z}(\tau) = z_0 + \int_0^{\tau} \bar{\alpha}(s) ds \in \overline{K}_{i_0}$, $\bar{z}_{i_0}(\tau) > L_{i_0} + R_0$ for some $\tau \leq T$ and $\bar{z}(s) \in \overline{\Omega}_p$ for all $s \in [0, \tau]$.
It is possible to extend $\bar{\alpha}$ by $s \mapsto \tilde{\alpha}(s - \tau)$ for $t - \tau > s > \tau$ where $\tilde{\alpha}$ is the control described in Assumption 2.1 for $i = i_0$, $t = t - \tau$ and $y_0 = \bar{z}_{0,i}^{\perp}(\tau)$.
It is easy to see that the related trajectory remains in $\overline{\Omega}_p$ and that there exists a constant C independent of z_0 in Case 2 such that

$$J(z_0, t, \bar{\alpha}) = \int_0^t D\psi_p(\bar{z}(s)) \cdot \bar{\alpha}(s) + \ell_0(\bar{z}(s), \bar{\alpha}(s)) ds \leq -t\overline{H}(O, p) + C,$$

which achieves the proof.

□

Lemma 5.2. *Let $p \in \mathbb{R}^N$ be such that $\overline{H}(O,p) > \Lambda$. There exists a constant $c \geq 0$ such that for all $z_0 \in \overline{\Omega}_p$ and $t \geq 0$,*

$$u(z_0,t) \geq -t\max(E, \overline{H}(O,p)) - c. \tag{5.10}$$

Proof. For any $\varepsilon > 0$, we consider $\alpha \in \mathcal{A}^p_{z_0}$ such that $J(z_0,t,\alpha) \leq u(z_0,t) + \varepsilon$. Call $z(s) = z_0 + \int_0^s \alpha(\theta)d\theta$. In what follows, c is a positive constant possibly depending on p that may change from line to line.

- Assume first that $z_0 \in \widetilde{K}_0$. We make out two cases:
  - If $z(t) \in \overline{\widetilde{K}}_0$, it is possible to modify the control law as in the Lemma 3.2 and Lemma 3.1 to obtain a $t+\tau$-periodic trajectory $z^*(\cdot)$. Then $s \mapsto \psi_p(z^*((s))$ is a function belonging to $W_0^{1,\infty}(0,t+\tau)$, whose weak derivative coincides almost everywhere with $s \mapsto D\psi_p(z^*(s)) \cdot \alpha^*(s)$. Therefore $\int_0^{t+\tau} D\psi_p(z^*(s)) \cdot \alpha^*(s)ds = \psi_p(z^*(t+\tau)) - \psi_p(z^*(0)) = 0$, then

    $$J(z_0, t+\tau, \alpha^*) = \int_0^{t+\tau} \ell_0(z^*(s), \alpha^*(s))ds \geq -Et - c.$$

    Thus,

    $$J(z_0, t+\tau, \alpha^*) = J(z_0,t,\alpha) + J(z(t), \tau, \alpha^*(\cdot + t)$$

    and $J(z(t), \tau, \alpha^*(\cdot + t)$ is bounded by a constant thanks to the definition of ψ_p for trajectories lying in $\widetilde{K}_0$.
    Therefore,

    $$J(z_0,t,\alpha) = J(z_0, t+\tau, \alpha^*) - J(z(t), \tau, \alpha^*(\cdot + t) \geq -Et - c.$$

  - $z(t) \in (\overline{K}_i \cup \overline{\widetilde{K}}_i) \backslash \overline{\widetilde{K}}_0$ for some $i \in \mathcal{I}(p)$. Let $\bar{t}$ be the supremum of all entry times in $\overline{K}_i \cup \overline{\widetilde{K}}_i$ smaller than t. Then $z(\bar{t}) \in \widetilde{K}_0$ and we can repeat the argument above to prove that

    $$J(z_0, \bar{t}, \alpha) \geq -E\bar{t} - c.$$

    There only remains to study $\int_{\bar{t}}^t (D\psi_p(z(s)) \cdot \alpha(s) + \ell_0(z(s), \alpha(s)))\, ds$. For $s \in [\bar{t}, t]$, $\ell_0(z(s), \alpha(s)) = \ell_i(0, z_i^\perp(s), \alpha(s))$. Moreover, $\left| \int_{\bar{t}}^t D\psi_p(z(s)) \cdot \alpha(s)ds - p_i(z_i(t) - z_i(\bar{t})) \right| \leq c$. Hence,

    $$\left| \int_{\bar{t}}^t (D\psi_p(z(s)) \cdot \alpha(s) + \ell_0(z(s), \alpha(s)))\, ds - \int_{\bar{t}}^t \Big(p_i\alpha_i(s) + \ell_i(0, z_i^\perp(s), \alpha(s)) \Big)\, ds \right| \leq c.$$

    But

    $$\int_{\bar{t}}^t \Big(p_i\alpha_i(s) + \ell_i(0, z_i^\perp(s), \alpha(s)) \Big)\, ds \geq v_i(0, p_i, z_i^\perp(\bar{t}), t - \bar{t})$$

    where v_i is defined in (2.8). From (2.9),

    $$\int_{\bar{t}}^t \Big(p_i\alpha_i(s) + \ell_i(0, z_i^\perp(s), \alpha(s)) \Big)\, ds \geq -(t-\bar{t})\overline{H}_i(0,p_i) - c. \tag{5.11}$$

    Summing all the contributions, we get that

    $$J(z_0,t,\alpha) \geq (t-\bar{t})(-\overline{H}_i(0,p_i)) - E\bar{t} - c.$$

    But $\overline{H}_i(0,p_i) = \overline{H}(O,p)$ since $i \in \mathcal{I}(p)$. Hence

    $$J(z_0,t,\alpha) \geq (t-\bar{t})(-\overline{H}(O,p)) - E\bar{t} - c \geq \min(-\overline{H}(O,p), -E)t - c.$$

- Assume now that $z_0 \in \overline{K}_i \cup \overline{\widetilde{K}}_i$ for some $i \in \mathcal{I}(p)$. We make out two cases:
  - If the trajectory stays in $\overline{K}_i \cup \overline{\widetilde{K}}_i$, then

$$J(z_0,t,\alpha) = \psi_p(z(t)) - \psi_p(z_0) + \int_0^t \ell_i(0, z_i^\perp(s), \alpha(s))ds$$
$$\geq p_i(z_i(t) - z_{0,i}) - c + \int_0^t \ell_i(0, z_i^\perp(s), \alpha(s))ds.$$

    Therefore, $J(z_0,t,\alpha) \geq v_i(0, z_{0,i}^\perp, t) - c \geq -t\overline{H}(0,p) - c$.
  - The trajectory z leaves $\overline{K}_i \cup \overline{\widetilde{K}}_i$. Let θ be the smallest time such that $z(\theta) \in \overline{\widetilde{K}}_0$. Then

$$J(z_0,t,\alpha) = J(z_0,\theta,\alpha) + J(z(\theta), t-\theta, \tilde{\alpha}),$$

    where $\tilde{\alpha}(s) = \alpha(s-\theta)$. We have just seen that $J(z_0,\theta,\alpha) \geq -\theta\overline{H}(O,p) - c$. From the previous case (when $z_0 \in \widetilde{K}_0$), $J(z(\theta), t-\theta, \tilde{\alpha}) \geq (t-\theta)\min(-\overline{H}(O,p), -E) - c$. Thus, $J(z_0,t,\alpha) \geq t\min(-\overline{H}(O,p), -E) - c$.

The result follows letting $\varepsilon \to 0$. $\square$

As a consequence of Lemmas 5.1 and 5.2, and using also that $E \geq \Lambda$ from Lemma 4.1, we obtain the following:

Lemma 5.3. *Let $p \in \mathbb{R}^N$ be such that $\overline{H}(O,p) > E$. Under Assumptions 1.1-1.3 and 2.1, the function v: $(z,t) \mapsto u(z,t) + t\overline{H}(O,p)$ is bounded and Lipschitz continuous on $\overline{\Omega}_p \times \mathbb{R}_+$. It is a viscosity solution of*

$$\frac{\partial v}{\partial t}(z,t) + H_0(z, D\psi_p + Dv) - \overline{H}(O,p) \geq 0 \qquad \text{in } \overline{\Omega}_p \times \mathbb{R}_+, \tag{5.12}$$
$$\frac{\partial v}{\partial t}(z,t) + H_0(z, D\psi_p + Dv) - \overline{H}(O,p) \leq 0 \qquad \text{in } \Omega_p \times \mathbb{R}_+. \tag{5.13}$$

Define the function w by

$$w(z,t) = \inf_{r>0} v(z, t+r).$$

It is easy to check that the function w is bounded and Lipschitz continuous. It can also be seen that w is a non decreasing function of time. From e.g. [9], Proposition 2.11, page 302, or [22], Proposition 1.10, page 125, w is a viscosity supersolution of

$$\frac{\partial w}{\partial t}(z,t) + H_0(z, D\psi_p + Dw) - \overline{H}(O,p) \geq 0 \quad \text{in } \overline{\Omega}_p \times \mathbb{R}_+.$$

From the convexity of $H_0(z,\cdot)$, we get the following Lemma

Lemma 5.4. *w is a viscosity subsolution of*

$$\frac{\partial w}{\partial t}(z,t) + H_0(z, D\psi_p + Dw) - \overline{H}(O,p) \leq 0 \quad \text{in } \Omega_p \times \mathbb{R}_+.$$

Proof. See [22] Lemma 5.11. Two possible arguments may be used, the first one from Barron and Jensen[13] , and the second one from Barles[10]. $\square$

Let $\chi_p(z)$ be defined by $\chi_p(z) = \lim_{t\to\infty} w(z,t) = \sup_t w(z,t)$. It is obvious that χ_p is bounded. The convergence of $w(z,t)$ to $\chi_p(z)$ is uniform on the bounded subsets of $\overline{\Omega}_p$. Therefore χ_p is a Lipschitz function on $\overline{\Omega}_p$. Hence, χ_p as a function of z and t is a viscosity solution of (5.12) and (5.13). Since it does not depend on t, χ_p is a bounded and Lipschitz solution of

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O,p) \geq 0 \qquad \text{in } \overline{\Omega}_p, \tag{5.14}$$

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O,p) \leq 0 \qquad \text{in } \Omega_p. \tag{5.15}$$

Therefore, the following theorem has been obtained:

Theorem 5.1. *Let $p \in \mathbb{R}^N$ be such that $\overline{H}(O,p) > E$. Under Assumptions 1.1-1.3 and 2.1, there exists a bounded and Lipschitz function χ_p defined on $\overline{\Omega}_p$ which is a viscosity solution of (5.14)-(5.15).*

6 Convergence

The aim is to study the asymptotic behavior of the sequence of functions (u_ε) as $\varepsilon \to 0+$ where u_ε is the unique viscosity solution of (1.5) (1.6).
A sequence of continuous functions (v_ε) defined on Ω_ε is said to converge locally uniformly to a continuous function v defined on $\mathcal{G}$ if for all $M > 0$,

$$\lim_{\varepsilon\to 0} \sup_{x\in\mathcal{G}, y\in\Omega_\varepsilon\cap B(O,M), |y-x|\leq\varepsilon} |v(x) - v_\varepsilon(y)| = 0.$$

Theorem 6.1. *Under Assumptions 1.1-1.3 and 2.1, consider the weak limits in the viscosity sense, or relaxed semilimits, of u_ε, as $\varepsilon \to 0+$: for all $x \in \mathcal{G}$,*

$$\underline{u}(x) = \liminf_{\varepsilon\to 0+, x'\to x, x'\in\Omega_\varepsilon} u_\varepsilon(x'); \quad \overline{u}(x) = \limsup_{\varepsilon\to 0+, x'\to x, x'\in\Omega_\varepsilon} u_\varepsilon(x').$$

Then, $\underline{u}$ is a bounded supersolution of

$$\begin{array}{lcll} \lambda\underline{u}(x) + \overline{H}(x, D\underline{u}(x)) & \geq & 0, & \text{if } x \in \mathcal{G}\backslash\{O\}, \\ \lambda\underline{u}(O) + \max(E, \overline{H}(O, D\underline{u}(O))) & \geq & 0, & \text{if } x = O, \end{array} \tag{6.1}$$

and $\overline{u}$ is a bounded subsolution of

$$\begin{array}{lcll} \lambda\overline{u}(x) + \overline{H}(x, D\overline{u}(x)) & \leq & 0, & \text{if } x \in \mathcal{G}\backslash\{O\}, \\ \lambda\overline{u}(O) + \max(E, \overline{H}(O, D\overline{u}(O))) & \leq & 0, & \text{if } x = O. \end{array} \tag{6.2}$$

Proof. Recall that u_ε is bounded by a constant M independent of ε from the assumptions on ℓ_ε. Therefore $\underline{u}$ and $\overline{u}$ are bounded. We first prove that $\overline{u}$ is a subsolution of (6.2).

$\overline{u}$ is a subsolution of (6.2) To prove that $\overline{u}$ is a subsolution of (6.2), we consider a strict maximum $\bar{x}$ of $\overline{u} - \varphi$, where φ is a regular test-function for (6.1)-(6.2). We may assume that $\varphi(\bar{x}) = \overline{u}(\bar{x})$.
We want to show that

$$\begin{array}{lcll} \lambda\varphi(\bar{x}) + \overline{H}(\bar{x}, D\varphi(\bar{x})) & \leq & 0, & \text{if } \bar{x} \in \mathcal{G}\backslash\{O\}, \\ \lambda\varphi(O) + \max(E, \overline{H}(O, D\varphi(O))) & \leq & 0, & \text{if } \bar{x} = O. \end{array} \tag{6.3}$$

We just have to focus on the case when $\bar{x} = O$, because if $\bar{x} \neq O$, (6.3) is proved in [3]. Suppose by contradiction that

$$\lambda\varphi(O) + \max(E, \overline{H}(O, D\varphi(O))) = \gamma > 0. \tag{6.4}$$

Let $p \in \mathbb{R}^N$ be defined by $p = (D\varphi(O, e_1), \ldots, D\varphi(O, e_N))$. We make out two cases:

First case: $\overline{H}(O,p) > E$**.** In this case (6.4) becomes:

$$\lambda\varphi(O) + \overline{H}(O, D\varphi(O)) = \gamma > 0.$$

For Ω_p defined in (5.2), let the auxiliary function ξ_ε be defined in $\varepsilon\overline{\Omega}_p$ by

$$\begin{array}{rcll} \xi_\varepsilon(x) &=& \varphi(x_i e_i) - \varphi(O) & \forall x \in \varepsilon\overline{K}_i,\ i \in \mathcal{I}(p), \\ \xi_\varepsilon(x) &=& \frac{\varepsilon}{x_i}(\varphi(x_i e_i) - \varphi(O))\Psi(\frac{x}{\varepsilon}) & \forall x \in \varepsilon\widetilde{K}_i,\ i \in \mathcal{I}(p), \\ \xi_\varepsilon(x) &=& 0 & \forall x \in \varepsilon\widetilde{K}_0, \end{array} \tag{6.5}$$

where Ψ is the same function as the one used in (5.1). Note that the function $x \mapsto \xi_\varepsilon(x) - \varepsilon\psi_p(\frac{x}{\varepsilon})$ is a $\mathcal{C}^1$ function which vanishes in $\varepsilon\widetilde{K}_0$ and that

$$\lim_{\rho\to 0} \sup_{0<\varepsilon<\frac{\rho}{R_0}} \|D\xi_\varepsilon(x) - D_z\psi_p(\frac{x}{\varepsilon})\|_{L^\infty(\varepsilon\overline{\Omega}_p \cap B_\rho(O))} = 0, \tag{6.6}$$

where R_0 is the number fixed at the beginning of § 5. Indeed,

$$\begin{array}{rcll} D\xi_\varepsilon(x) &=& \frac{d\varphi}{dx_i}(x_i e_i)e_i = p_i e_i + o(1) = D_z\psi_p(\frac{x}{\varepsilon}) + o(1) & \forall x \in \varepsilon\overline{K}_i, \\ D\xi_\varepsilon(x) &=& \frac{\varphi(x_i e_i) - \varphi(O)}{x_i}\frac{\partial\Psi}{\partial z_i}(\frac{x}{\varepsilon})e_i - \frac{\varepsilon}{x_i}\left(\frac{\varphi(x_i e_i)-\varphi(O)}{x_i} - \frac{d\varphi}{dx_i}(x_i e_i)\right)\Psi(\frac{x}{\varepsilon})e_i & \\ &=& (p_i + o(1))\frac{\partial\Psi}{\partial z_i}(\frac{x}{\varepsilon})e_i + \frac{\varepsilon}{x_i}\Psi(\frac{x}{\varepsilon})o(1)e_i & \\ &=& D_z\psi_p(\frac{x}{\varepsilon}) + o(1) & \forall x \in \varepsilon\widetilde{K}_i, \\ D\xi_\varepsilon(x) &=& 0 & \forall x \in \varepsilon\widetilde{K}_0, \end{array}$$

where $o(1)$ means a family of functions $(f_\varepsilon)_\varepsilon$ defined in $\varepsilon\overline{\Omega}_p$ such that

$$\lim_{\rho\to 0} \sup_{0<\varepsilon<\frac{\rho}{R_0}} \|f_\varepsilon\|_{L^\infty(\varepsilon\overline{\Omega}_p \cap B_\rho(O))} = 0.$$

Define the perturbed test-function

$$\varphi_\varepsilon(x) = \varphi(O) + \xi_\varepsilon(x) + \varepsilon\chi_p(\frac{x}{\varepsilon})$$

in $\varepsilon\overline{\Omega}_p$, where χ_p is constructed at the end of § 5 and is a bounded viscosity solution of (5.14), (5.15).
We claim that for some $\rho > 0$, φ_ε is a viscosity supersolution of

$$\lambda\varphi_\varepsilon(x) + H_\varepsilon(x, D\varphi_\varepsilon(x)) \geq 0 \quad x \in \varepsilon\overline{\Omega}_p \cap B_\rho(O), \tag{6.7}$$

for all ε such that $\varepsilon < \rho/R_0$.
Consider a smooth function ζ such that $\varphi_\varepsilon - \zeta$ attains its minimum in $\varepsilon\overline{\Omega}_p \cap B_\rho(O)$ at $\tilde{x}$ and $\varphi_\varepsilon(\tilde{x}) = \zeta(\tilde{x})$. Then the function $z \mapsto \chi_p(z) + \psi_p(z) - \left(\psi_p(z) - \frac{1}{\varepsilon}\xi_\varepsilon(\varepsilon z) + \frac{1}{\varepsilon}\zeta(\varepsilon z)\right)$ attains its minimum at $\tilde{z} = \frac{\tilde{x}}{\varepsilon}$. From the definition of χ_p we get

$$H_0\left(\tilde{z}, D\zeta(\tilde{x}) - D\xi_\varepsilon(\tilde{x}) + D_z\psi_p(\frac{\tilde{x}}{\varepsilon})\right) \geq \overline{H}(O,p)$$

and therefore,

$$\lambda\varphi(O) + H_0\left(\tilde{z}, D\zeta(\tilde{x}) - D\xi_\varepsilon(\tilde{x}) + D_z\psi_p(\frac{\tilde{x}}{\varepsilon})\right) \geq \lambda\varphi(O) + \overline{H}(O,p) = \gamma > 0.$$

But

$$\zeta(\tilde{x}) - \varphi(O) = \xi_\varepsilon(\tilde{x}) + \varepsilon\chi_p\left(\frac{\tilde{x}}{\varepsilon}\right).$$

From (6.5) and (6.6), it is possible to choose ρ small enough such that

1. $\lambda \sup_{0<\varepsilon<\frac{\rho}{R_0}} \max_{x\in\varepsilon\overline{\Omega}_p\cap B_\rho(O)} |\xi_\varepsilon(x)| \leq \frac{\gamma}{6}$
2. $\lambda \sup_{0<\varepsilon<\frac{\rho}{R_0}} \max_{x\in\varepsilon\overline{\Omega}_p\cap B_\rho(O)} |\varepsilon\chi_p\left(\frac{x}{\varepsilon}\right)| \leq \frac{\gamma}{6}$
3. $\sup_{0<\varepsilon<\frac{\rho}{R_0}} \max_{x\in\varepsilon\overline{\Omega}_p\cap B_\rho(O)} \left|H_0\left(\frac{x}{\varepsilon}, D\zeta(x) - D\xi_\varepsilon(x) + D_z\psi_p(\frac{x}{\varepsilon})\right) - H_0\left(\frac{x}{\varepsilon}, D\zeta(x)\right)\right| \leq \frac{\gamma}{6}$

We get that

$$\lambda\zeta(\tilde{x}) + H_0(\tilde{z}, D\zeta(\tilde{x})) \geq \frac{\gamma}{2}.$$

Moreover, from the definition of ℓ_ε, the continuity of the functions ℓ_i and the compactness of the set $[-1,1]\times A$, we see that

$$\lim_{\rho\to 0} \sup_{0<\varepsilon<\frac{\rho}{R_0}} \|H_0(\frac{x}{\varepsilon}, p) - H_\varepsilon(x,p)\|_{L^\infty(\varepsilon\overline{\Omega}_p\cap B_\rho(O))} = 0. \tag{6.8}$$

Therefore, for ρ small enough,

$$\lambda\zeta(\tilde{x}) + H_\varepsilon(\tilde{x}, D\zeta(\tilde{x})) \geq \frac{\gamma}{4},$$

which proves the claim.
From (1.5) and (6.7), we deduce that

$$\sup_{\varepsilon\overline{\Omega}_p\cap B_\rho(O)} u_\varepsilon - \varphi_\varepsilon \leq \sup_{\varepsilon\Omega_p\cap\partial B_\rho(O)} u_\varepsilon - \varphi_\varepsilon.$$

Passing to the limit when $\varepsilon\to 0$, we obtain

$$0 = \bar{u}(O) - \varphi(O) \leq \bar{u}(\rho e_i) - \varphi(\rho e_i), \quad \forall i\in\mathcal{I}(p).$$

Since we can choose ρ as small as we like, we reach a contradiction because O is a strict maximum of $\bar{u}-\varphi$.

Second case: $\overline{H}(O,p)\leq E$. In this case (6.4) becomes:

$$\lambda\varphi(O) + E = \gamma > 0.$$

Let us modify φ by taking $\tilde{\varphi}(x) = \varphi(O) + M|x|$. We choose M large enough such that

- $M \geq q$ where q is defined in Remark 3.2
- the function $\tilde{\varphi}$ is still an admissible test-function such that $\bar{u}-\tilde{\varphi}$ has a strict local maximum at O and $\overline{H}(O, D\tilde{\varphi}(O)) \leq \overline{H}(O,p)$, so

$$\max(E, \overline{H}(O, D\tilde{\varphi}(O))) \leq \max(E, \overline{H}(O,p)) = E.$$

Therefore, (6.4) holds for $\tilde{\varphi}$.

For simplicity, we agree to write $\varphi = \tilde{\varphi}$.
Thanks to Remark 3.2, there exists R^* such that the function $\tilde{w}^{R^*}(z)$ defined in (3.12) (recall that $M \geq q$) is a viscosity supersolution of

$$H_0(z, D\tilde{w}^{R^*}(z)) \geq E - \frac{\gamma}{4} \quad \text{for any } z \in \bar{\Omega}.$$

Note that, with z^{R^*} defined in (3.12), if $z \in \bar{\Omega} \backslash \bar{\Omega}^{R^*}$ then $\tilde{w}^{R^*}(z) - \varphi(z) = \tilde{w}^{R^*}(z^{R^*}) - \varphi(O) - MR^*$.
Let us call φ_ε^* the function defined in $\varepsilon\overline{\Omega}$ by $\varphi_\varepsilon^*(x) = \varphi(O) + \varepsilon \tilde{w}^{R^*}(\frac{x}{\varepsilon})$,
then if $\frac{x}{\varepsilon} \in \bar{\Omega} \backslash \bar{\Omega}^{R^*}$

$$\varphi_\varepsilon^*(x) - \varphi(x) = \varepsilon \left(\tilde{w}^{R^*}(\frac{x}{\varepsilon}) - M\frac{|x|}{\varepsilon} \right) = O(\varepsilon).$$

For $\rho > 0$ smaller than a constant independent of ε, we claim that φ_ε^* is a viscosity supersolution of

$$\lambda \varphi_\varepsilon^*(x) + H_\varepsilon(x, D\varphi_\varepsilon^*(x)) \geq O \quad x \in \varepsilon\overline{\Omega} \cap \overline{B_\rho(O)},$$

for all $\varepsilon > 0$. Let ζ a regular function and $x^* \in \varepsilon\overline{\Omega} \cap \overline{B_\rho(O)}$ be such that

$$\varphi_\varepsilon^*(x) - \zeta(x) \geq \varphi_\varepsilon^*(x^*) - \zeta(x^*) = 0, \quad x \in \varepsilon\overline{\Omega} \cap \overline{B_\rho(O)},$$

This implies that, changing the variable $z = \frac{x}{\varepsilon}$ and $z^* = \frac{x^*}{\varepsilon}$

$$\tilde{w}^{R^*}(z) - \frac{1}{\varepsilon}\zeta(\varepsilon z) \geq \tilde{w}^{R^*}(z^*) - \frac{1}{\varepsilon}\zeta(\varepsilon z^*)$$

It is possible to use the equation satisfied by w^{R^*}:

$$H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) \geq E - \frac{\gamma}{4}.$$

If $z^* = \frac{x^*}{\varepsilon} \in \overline{\Omega}^{R^*}$

$$\begin{aligned}
\lambda \varphi_\varepsilon^*(x^*) + H_\varepsilon(x^*, D\zeta(x^*)) &= \lambda \varphi_\varepsilon^*(x^*) + H_\varepsilon(x^*, D\zeta(x^*)) - H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) + H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) \\
&\geq \lambda\varphi(0) - C(R^*)\varepsilon + H_\varepsilon(x^*, D\zeta(x^*)) - H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) + H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) \\
&\geq -E + \gamma + (E - \frac{\gamma}{4}) - C(R^*)\varepsilon + H_\varepsilon(x^*, D\zeta(x^*)) - H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) \\
&= \frac{3}{4}\gamma + O(\varepsilon) + H_\varepsilon(x^*, D\zeta(x^*)) - H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) \\
&= \frac{1}{2}\gamma + H_\varepsilon(x^*, D\zeta(x^*)) - H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*))
\end{aligned} \tag{6.9}$$

If $z^* = \frac{x^*}{\varepsilon} \in \overline{\Omega} \backslash \overline{\Omega}^{R^*}$

$$\lambda \varphi_\varepsilon^*(x^*) + H_\varepsilon(x^*, D\zeta(x^*)) \geq -C\rho + \frac{1}{2}\gamma + H_\varepsilon(x^*, D\zeta(x^*)) - H_0(\frac{x^*}{\varepsilon}, D\zeta(x^*)) \tag{6.10}$$

Using an argument similar to that used to prove (6.8) for ρ small enough we can deduce from (6.9) and (6.10)

$$\lambda \varphi_\varepsilon^*(x^*) + H_\varepsilon(x^*, D\zeta(x^*)) \geq 0. \tag{6.11}$$

We deduce from this that

$$\sup_{\varepsilon\overline{\Omega} \cap B_\rho(O)} u_\varepsilon - \varphi_\varepsilon^* \leq \sup_{\varepsilon\Omega \cap \partial B_\rho(O)} u_\varepsilon - \varphi_\varepsilon^*$$

which leads to a contradiction as above.

$\underline{u}$ **is a supersolution of (6.1)** We are left with proving that $\underline{u}$ is a supersolution of (6.1). We consider a strict minimum $\bar{x}$ of $\underline{u} - \tilde{\varphi}$, such that $\underline{u}(\bar{x}) = \tilde{\varphi}(\bar{x})$, where $\tilde{\varphi}$ is a regular test-function for (6.2)-(6.1). In the case when $\bar{x} \neq O$, the desired result is proved in [3]. Thus, we may focus on the case when $\bar{x} = O$.
We want to show that

$$\lambda\tilde{\varphi}(O) + \max(E, \overline{H}(O, D\tilde{\varphi}(O))) \geq 0. \tag{6.12}$$

We suppose by contradiction that

$$\lambda\tilde{\varphi}(O) + \max(E, \overline{H}(O, D\tilde{\varphi}(O))) = \tilde{\gamma} < 0.$$

Let $\tilde{p} \in \mathbb{R}_N$ be defined by $\tilde{p} = (D\tilde{\varphi}(O, e_1), \dots, D\tilde{\varphi}(O, e_N))$

First case If $\overline{H}(O, \tilde{p}) > E$ and $\mathcal{I}(\tilde{p}) = \{1, \dots, N\}$, i.e. for all $i \in \{1, \dots, N\}$, $\overline{H}_i^+(O, D\tilde{\varphi}(0, e_i)) = \overline{H}(O, \tilde{p})$, then we choose $\varphi = \tilde{\varphi}$, and $p = \tilde{p}$.

Second case On the contrary, i.e. if either $\overline{H}(O, \tilde{p}) \leq E$ or $\overline{H}(O, \tilde{p}) > E$ and $\mathcal{I}(\tilde{p}) \neq \{1, \dots, N\}$, it is possible to add a piecewise linear nonpositive function vanishing at O to $\tilde{\varphi}$ and obtain a new function φ such that

- O is a strict minimum of $\underline{u} - \varphi$, and $\underline{u}(O) = \varphi(O)$
- if $p = (D\varphi(O, e_1), \dots, D\varphi(O, e_N))$, then $\lambda\varphi(O) + \overline{H}(O, p) = \gamma < 0$
- $\overline{H}(O, p) > E$ and $\mathcal{I}(p) = \{1, \dots, N\}$, i.e. for all $i \in \{1, \dots, N\}$, $\overline{H}_i^+(O, p_i) = \overline{H}(O, p)$.

We then consider ψ_p as in (5.1), χ_p defined at the end of §5 with $\overline{\Omega}_p = \overline{\Omega}$, and ξ_ε as above. The perturbed test function is defined in $\varepsilon\Omega$ by

$$\varphi_\varepsilon(x) = \varphi(O) + \xi_\varepsilon(x) + \varepsilon\chi_p(\frac{x}{\varepsilon}).$$

We claim that for some $\rho > 0$, φ_ε is a viscosity subsolution of

$$\lambda\varphi_\varepsilon(x) + H_\varepsilon(x, D\varphi_\varepsilon(x)) \leq 0 \quad x \in \varepsilon\Omega \cap B_\rho(O), \tag{6.13}$$

for all ε such that $R_0\varepsilon < \rho$. The proof of the claim is essentially the same as for (6.7).
The comparison principle implies that

$$\inf_{\varepsilon\overline{\Omega}\cap B_\rho(O)} u_\varepsilon - \varphi_\varepsilon \geq \inf_{\varepsilon\Omega\cap\partial B_\rho(O)} u_\varepsilon - \varphi_\varepsilon.$$

and then

$$0 = \underline{u}(O) - \varphi(O) \geq \inf_{\mathcal{G}\cap\partial B_\rho(O)} \underline{u} - \varphi.$$

Since we can choose ρ as small as we like, we reach a contradiction because O is a strict minimum of $\underline{u} - \varphi$. □

Proof of Theorem 4.2. From Theorem 6.1, the relaxed semilimits $\bar{u}$ and $\underline{u}$ are respectively bounded viscosity subsolution and supersolution of (4.10)-(4.11). By comparison (Theorem 4.1), this implies that $\bar{u} = \underline{u} = u$ and the local uniform convergence of u_ε to u. □

A Proofs of Propositions 2.1 and 2.2

Proof of Proposition 2.1 Let us prove that Assumption 2.1 holds with $L_i = 0$.
Consider first the case when $p_i < \underline{p}_i$: from the convexity assumptions, there exists $\bar{a} \in A$ and $\bar{y} \in [-1, 1]$ such that $\bar{a}_i^\perp = 0$ and

$$\overline{H}_i(0, p_i) = -p_i \bar{a}_i - \ell_i(0, \bar{y}, \bar{a}).$$

Since $\overline{H}_i(0, p_i) > \Lambda_i^0$, we know that $\bar{a}_i \geq 0$. With the constant r appearing in Assumption 1.1, if $\tilde{t} = \frac{|\bar{y}-y_0|}{r}$, the control law $\tilde{\alpha}$ defined by

$$\begin{aligned} \tilde{\alpha}(s) &= r \operatorname{sign}(\bar{y} - y_0) e_i^\perp \quad \text{if } 0 < s < \tilde{t}, \\ \tilde{\alpha}(s) &= \bar{a} \quad \text{if } \tilde{t} < s, \end{aligned}$$

belongs to $\mathcal{A}_{i,y_0}$. Moreover $\int_0^t \tilde{\alpha}_i(s) ds = \bar{a}_i (t - \tilde{t})_+ \geq 0$. It is clear that if $t \geq 2/r$, then $t \geq \tilde{t}$ and

$$\begin{aligned} \int_0^t (p_i \tilde{\alpha}_i(s) + \ell_i(0, \tilde{y}(s), \tilde{\alpha}(s))) \, ds &\leq -(t - \tilde{t}) \overline{H}_i(0, p_i) + \frac{2}{r} \max_{(y,a) \in [-1,1] \times A} \ell_i(0, y, a) \\ &\leq -t \overline{H}_i(0, p_i) + c, \end{aligned}$$

for a constant c possibly depending on p_i and $\tilde{y}(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s) ds$. With C as in (2.9), we deduce that

$$\int_0^t (p_i \tilde{\alpha}_i(s) + \ell_i(0, \tilde{y}(s), \tilde{\alpha}(s))) \, ds \leq v_i(0, p_i, y_0, t) + C + c.$$

It is always possible to choose $C_i \geq C + c$ such that for all $0 \leq t \leq 2/r$, there is an admissible control law such that $\int_0^s \tilde{\alpha}_i(\tau) d\tau \geq 0, \quad \forall 0 \leq s \leq t$ and (2.19) holds.

We are left with the case when $p_i \in \operatorname{argmin} \overline{H}_i(0, \cdot)$: in that case, there exists $\bar{y} \in [-1, 1]$ such that

$$\overline{H}_i(0, p_i) = -\ell_i(0, \bar{y}, 0).$$

We can repeat the arguments above with $\bar{a} = 0$ and prove the claim.

Proof of Proposition 2.2 For brevity, we only discuss the case when $p_i < \underline{p}_i$, because the case $p_i \in [\underline{p}_i, \overline{p}_i]$ is handled in the same manner.
By adding the constant Λ_i^0 to ℓ_i, it is always possible to assume that $\Lambda_i^0 = 0$ and

$$\underline{c}_i |a|^{\nu_i} \leq \ell_i(0, y, a) \leq \bar{c}_i |a|^{\nu_i}. \tag{A.1}$$

Thus, $0 \in [\underline{p}_i, \overline{p}_i]$.

Step 1 Consider $y_0 \in [-1, 1]$, and $(\alpha, T) \in \mathcal{A}_{i,y_0} \times \mathbb{R}_+$ such that $\min_{0 \leq \theta \leq T} \int_0^\theta \alpha_i(s) ds \leq -L_i$. Let θ be such that $\int_0^\theta \alpha_i(s) ds \leq -L_i$. Hölder's inequality yields

$$L_i \leq \int_0^\theta \alpha_i^-(s) ds \leq \theta^{1 - \frac{1}{\nu_i}} \left(\int_0^\theta |\alpha(s)|^{\nu_i} ds \right)^{\frac{1}{\nu_i}},$$

thus

$$\int_0^\theta |\alpha(s)|^{\nu_i} ds \geq L_i^{\nu_i} \theta^{1 - \nu_i}.$$

Hence, any α such that

$$\int_0^T \alpha_i(s)ds = 0, \tag{A.2}$$

$$\min_{0\le\theta\le T}\int_0^\theta \alpha_i(s)ds \le -L_i \tag{A.3}$$

satisfies

$$\int_0^T \ell_i(0, y(s), \alpha(s))ds \ge \underline{c}_i L_i^{\nu_i} T^{1-\nu_i}, \tag{A.4}$$

with

$$y(s) = y_0 + \int_0^s \alpha_i^\perp(\tau)d\tau.$$

Moreover, if $L_i > \frac{2\max_A a_i^-}{r}$, where r is the constant in Assumption 1.1, then (A.3) implies that $T > \frac{2}{r}$. Therefore, from Assumption 1.1, the control

$$\bar{\alpha} = \bar{\alpha}_i^\perp e_i^\perp = \frac{y(T) - y_0}{T} e_i^\perp = \frac{1}{T}\int_0^T \alpha(\tau)d\tau \tag{A.5}$$

belongs to A.
On the other hand, with $\bar{\alpha}$ defined by (A.5), (2.23) implies that

$$\int_0^T \ell_i(0, y_0 + \bar{\alpha}_i^\perp s, \bar{\alpha})ds \le \bar{c}_i 2^{\nu_i} T^{1-\nu_i}. \tag{A.6}$$

Hence, from (A.4) and (A.6), it is possible to choose L_i large enough, namely larger than $\max\left(\frac{2\max_A a_i^-}{r}, 2\left(\frac{\bar{c}_i}{\underline{c}_i}\right)^{\frac{1}{\nu_i}}\right)$, such that for all (T, α) satisfying (A.2) (A.3), $\bar{\alpha}$ defined by (A.5) is an admissible control in $[0, T]$ and the following inequality holds

$$\int_0^T \ell_i(0, y(s), \alpha(s))ds \ge \int_0^T \ell_i(0, y_0 + \bar{\alpha}_i^\perp s, \bar{\alpha})ds. \tag{A.7}$$

Step 2 Take L_i as in the conclusion of Step 1 and p_i such that $p_i < \underline{p}_i$. Therefore $p_i < 0$. Since $\Lambda_i^0 = 0$, we also have that $\overline{H}_i(0, p_i) > 0$. Take C as in (2.9): for $t > \frac{2C}{\overline{H}_i(0,p_i)}$, $v_i(0, p_i, y_0, t) < -C$. There exists a control law $\alpha \in \mathcal{A}_{i,y_0}$ such that

$$\int_0^t (p_i\alpha_i(s) + \ell_i(0, y(s), \alpha(s)))\, ds \le v_i(0, p_i, y_0, t) + C < 0,$$

where y is given by (2.1). Then, $\int_0^t \alpha_i(s)ds > 0$. If $\min_{0\le s\le t}\int_0^s \alpha_i(\tau)d\tau \le -L_i$, we see from Step 1 that it is possible to replace α by a control $\tilde{\alpha}$ satisfying (2.18) and such that

$$\int_0^t (p_i\tilde{\alpha}_i(s) + \ell_i(0, \tilde{y}(s), \tilde{\alpha}(s)))\, ds \le \int_0^t (p_i\alpha_i(s) + \ell_i(0, y(s), \alpha(s)))\, ds,$$

hence (2.19) with $C_i = C$, and $\tilde{y}(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s)ds$.
It is always possible to choose $C_i \ge C$ such that for all $t \le \frac{2C}{\overline{H}_i(0,p_i)}$, there exists an admissible control law such that (2.18) and (2.19) hold.

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